to complex values they admit of similar expansions in the complex plane. This beautiful generalization of the theorems of Taylor and Maclaurin was one of the important discoveries due to Cauchy.

In order for a function to have a power series development in the neighborhood of a point it must possess derivatives of all orders at that point. While the class of functions that has this property is a very important one, it is nevertheless a highly restricted group when compared to functions in general. More general expansions are obviously necessary for the study of wider classes of functions. Moreover, even in the case of functions that possess a power series development, expansions in terms of other simple functions may be more useful for certain types of investigation.

Of coordinate importance with power series and next to them in simplicity we have Fourier series or developments in sines and cosines of integral multiples of x. These developments were first studied in connection with certain problems of mathematical physics and they still rank as the expansions of greatest utility in the application of mathematics to other sciences. This arises from the frequent occurrence of periodic phenomena and the importance of wave motion as a hypothesis in explaining other phenomena.

Among the important problems of mathematical physics whose solution frequently involves Fourier series may be mentioned those that arise in the study of the flow of heat and electricity. Such problems also lead to the consideration of other related expansions, such as developments in Bessel's functions, Laplace's functions, Sturm-Liouville functions, and so on. These various functions have in common with the sine and cosine the properties of being oscillatory in character and satisfying differential equations of the second order. The expansions in terms of such functions have many properties analogous to those of Fourier series, but their study naturally presents greater difficulty.

Outside of power series, Fourier series and related developments, the type of series that has been most extensively studied is that known as Dirichlet's series. Such series have the form $\Sigma a_n e^{-\lambda_n s}$, where the λ's are a set of real constants such that λ_n becomes infinite with n, the a's are a set of constants real or complex, and s is a complex variable. So far the applications of such series have been primarily in the field of pure mathematics, and more particularly in the analytic study of the Theory of Numbers. However, like other conceptions[1] that first arose in the study of pure mathematics, they may prove later to have important applications in other sciences.

In order to apply mathematical theory to concrete situations we need always a method of obtaining numerical results. Thus in the case of an infinite series of constant terms we must have some convention as to a numerical value to be attached to the series. The simplest and most natural one is based on the conception known as convergence. Given a series

$$u_0 + u_1 + u_2 + \cdots + u_n + \cdots, \tag{1}$$

[1] Such as conic sections, non-Euclidean geometry, complex numbers, continued fractions.

we form

$$s_n = u_0 + u_1 + \cdots + u_n \tag{2}$$

and consider the behavior of s_n with increasing n. If s_n approaches a definite limit s as n becomes infinite, we say that the series converges to the value s. If no limit exists we say that the series diverges. A series may diverge because the related s_n becomes positively or negatively infinite, or because it oscillates between finite or infinite limits. The latter type of divergent series is sometimes referred to as oscillatory.

2. *Use of divergent series in the eighteenth century.* The mathematicians of the eighteenth century found that the formal expansions of many simple functions failed to converge for values of the argument for which the function corresponding to the series was perfectly well defined. They found on the other hand that in many of these cases the divergent expansions could be used in various ways to obtain valid results just as well as if they were convergent. They therefore felt the need of attaching a numerical value to a divergent series that would be in agreement with the value of the function whose expansion gave rise to the series.

The procedure of two of the leading mathematicians of the period can be adequately illustrated in terms of the simple divergent series

$$1 - 1 + 1 - 1 + \cdots .$$

We note that the successive values of s_n are alternately 1 and 0. Since these two values occur with equal frequency, Leibnitz contended that on the basis of the Theory of Probability the value $\frac{1}{2}$ should be ascribed to the series. Suppose, on the other hand, that we introduce successive powers of x in the terms of the series, thus obtaining the power series

$$1 - x + x^2 - x^3 + \cdots .$$

For values of x numerically less than unity the series converges and has the value $1/(1+x)$. Now allow x to approach 1 from below. The function defined by the series approaches $\frac{1}{2}$; the series itself, if we take the limit term by term, approaches the series with which we started. For this reason Euler contended that the value $\frac{1}{2}$ should be ascribed to the original series. The surprising feature here is that these two entirely different methods of approach lead to the same value for the series. This coincidence is not an accident, however, but a special case of a general theorem proved by Frobenius in 1880, to which we will refer later.

It is apparent that the procedure of Leibnitz and Euler in the case of the simple series $1-1+1-1+\cdots$ is entirely out of harmony with present day notions of rigor in mathematical analysis. These notions are the result of a gradual development from ideas that first came into prominence in the mathematical world early in the nineteenth century as a consequence of the pronouncements of such outstanding mathematicians as Cauchy and Abel. The

steadily rising standards of rigor, coupled with the fact that the use of divergent series occasionally led to serious contradictions, resulted in a gradually diminishing interest in such series during the first three fourths of the nineteenth century. At the same time the rigorous theory of convergent series was being actively developed by students of this field of mathematics, and thus the way was being paved for a rigorous treatment of divergent series.

3. *Beginnings of the modern theory.* The first definite advance in the direction of a rigorous theory to appear in a mathematical journal is found in a paper by Frobenius which was published in Crelle's Journal in 1880. The essential contribution of Frobenius was the proof of the following theorem.

If for the series (1) *the* s_n *defined by* (2) *are such that the limit,* $\lim_{n\to\infty}(s_0+s_1+\cdots+s_n)/(n+1)$, *exists and is equal to* s, *then the series* $u_0+u_1x+\cdots+u_nx^n+\cdots$ *will converge for* $|x|<1$ *to a function* $f(x)$ *such that* $\lim_{x\to 1-0}f(x)=s$. It is apparent that the agreement between the procedures of Leibnitz and Euler in the case of the series $1-1+1-1+\cdots$ is a special case of Frobenius's theorem.

Two years after the publication of Frobenius's paper, his result was considerably generalized by Hölder. Let us represent the successive arithmetic means of the s's by $s_0^{(1)}, s_1^{(1)}, \cdots s_n^{(1)}, \cdots$. Then, if $s_n^{(1)}$ fails to approach a limit as n becomes infinite, it seems natural to repeat the process originally used by forming successive arithmetic means of the $s^{(1)}$'s, labeling them $s_0^{(2)}, s_1^{(2)}, \cdots s_n^{(2)}, \cdots$. If $s_n^{(2)}$ approaches a limit s we define that limit to be the generalized sum of the series. If $s_n^{(2)}$ does not approach a limit, we repeat the process, forming $s^{(3)}$'s and studying the behavior of $s_n^{(3)}$ as n becomes infinite. If for any integer r, $s_n^{(r)}$ approaches a limit s as n becomes infinite, we define that limit to be the generalized sum of the original series. In modern terminology we say that the series is summable (Hr) to the value s. We are now in a position to state Hölder's generalization of Frobenius's theorem. *If the series* (1) *is summable* (Hr) *to the value* s *for any integer* r, *then the series* Σu_nx^n *will converge for values of* x *numerically less than* 1 *to a function* $f(x)$ *such that* $\lim_{x\to 1-0}f(x)=s$.

In 1890, ten years after the publication of Frobenius's paper, Cesàro attacked a problem in series that is naturally suggested by the behavior of the Cauchy product of two convergent series. Given two infinite series, Σu_n and Σv_n, their Cauchy product is defined as the series $\Sigma(u_nv_0+u_{n-1}v_1+\cdots+u_0v_n)$. If both the original series are convergent and at least one of them is absolutely convergent, it can be shown that the product series will always converge to a value that is the product of the values of the two given series. If, however, neither of the original series is absolutely convergent, it may happen that the product series diverges. If then there can be found methods of summing divergent series that will give to them values consistent with their mode of formation, it seems reasonable to expect that some one of these methods will serve to sum the product series of two convergent series to the proper value. Cesàro found that the simple method used by Frobenius had this interesting and important property. However, the problem suggested by the multiplication of

convergent series was not completely elucidated by this discovery, nor did Cesàro's investigations stop at that point. Suppose that we are considering the product of three series. The product of two of them may diverge, and we are thus led to consider the behavior of the product of a convergent series and a series that can be summed by the method of Frobenius. If we start with four series, their products, taking two at a time, may each diverge, and we thus have the problem of multiplying two divergent series of a certain type, and so on. It was Cesàro's aim to answer all questions of this order, and thus to furnish the means for discussing the behavior of the product of any finite number of series. He succeeded brilliantly in his purpose, obtaining a complete solution of the problem in a six page paper that involved only analysis of a simple type.

Cesàro arrived at his goal by generalizing the process of forming an arithmetic mean of the first n partial sums of the series in a different manner from that adopted by Hölder. Instead of forming successive means from the means previously obtained, he conceived the idea of forming successive weighted means, the weights changing in a regular manner. Let us set

$$(4)\quad S_n^{(r)} = s_n + rs_{n-1} + \frac{r(r+1)}{2!}s_{n-2} + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!}s_0,$$

$$(5)\quad A_n^{(r)} = \frac{(r+1)(r+2)\cdots(r+n)}{n!}$$

$$(6)\quad \sigma_n^{(r)} = \frac{S_n^{(r)}}{A_n^{(r)}}.$$

The $A_n^{(r)}$ can be readily shown to be the sum of the coefficients of the s's in the expression for $S_n^{(r)}$, and therefore $\sigma_n^{(r)}$ is a weighted mean of $s_0, s_1, \cdots, s_n$. For $r=0$, $\sigma_n^0 = s_n$, and consequently convergence is included in our general scheme of summation for this value of the parameter. For $r=1$, $\sigma_n^{(1)} = \sum_{i=0}^{i=n} s_i/(n+1)$, and therefore summation by the ordinary arithmetic mean is included for the value unity of the parameter. For values of $r>1$, it is apparent from the above formula that the early s's are weighted more than the later ones, and therefore it seems probable that series which oscillate more strongly can be summed by using larger values of r. Since the general effect of forming the Cauchy product of two or more oscillating series is to increase the degree of oscillation, one would expect the general multiplication theorem to depend on the value of r in formula (6), and this expectation is realized. In modern terminology a series for which $\sigma_n^{(r)}$ approaches a limit σ as n becomes infinite is said to be summable (Cr). Using this definition, Cesàro's general theorem may be stated as follows. *The Cauchy product of a series summable* (Cr) *to value* A *by a series summable* (Cs) *to value* B *is always summable* $(C, r+s+1)$ *to value* AB. This includes the result regarding convergent series as the special case where $r=0=s$.

It is apparent at first sight that there is a rather close relationship between Hölder's generalization of Frobenius's method and that of Cesàro. Furthermore

we readily find that a necessary condition for either summability (Hr) or (Cr) is $\lim_{n\to\infty} u_n/n^r = 0$. This suggests that the two methods are of the same general scope, and mathematicians working with them felt fairly certain that they were either equivalent or approximately so. Nevertheless, the discussion of the question of equivalence seemed at first to involve algebraic difficulties of a serious character. Finally, in 1907, Knopp proved that summability (Hr) implies summability (Cr) to the same value, and somewhat later Schur and W. B. Ford proved independently that summability (Cr) implies summability (Hr) to the same value. These early proofs involved rather complicated analysis, but it has since been found that the equivalence theorem can be proved in a quite elementary manner. The first simple proof was given by Schur in 1913.

4. *Work of Borel and others on analytic extension.* We spoke at the beginning of the importance of developments in power series in the case of functions possessing such developments. For some functions, such as the sine and cosine, the corresponding power series converges for all values of the independent variable. In the case of other quite simple functions, however, the range of convergence of the power series may be quite limited. For example the power series development of the function $1/(1-z)$ is

$$1 + z + z^2 + \cdots + z^n + \cdots, \tag{7}$$

and this series converges only within the unit circle. The Theory of Functions of a Complex Variable furnishes the reason for this restricted region of convergence. The function $1/(1-z)$, simple as it is, possesses an infinite discontinuity at the point $z=1$, and such a discontinuity on the circumference of a circle whose center is the point about which we develop the function always prevents convergence outside the circle. In spite of the failure of the convergence of the series for values of $z \geqq 1$, there is obviously such a close relationship between the series and the function that we should expect to find the value of the function for such values of z by using a suitable method of summing the divergent series. Cesàro's success in summing the product of any finite number of convergent series suggested to Borel the possibility of summing power series outside their circle of convergence by using some similar scheme. It was apparent at once that neither the method of Cesàro nor that of Hölder would suffice, since the necessary condition for their applicability is not satisfied by a power series beyond the circle of convergence.

Borel found it useful, however, to take Cesàro's scheme of weighted means as a point of departure, but he introduced an important new element in this scheme. Instead of considering a weighted mean of a finite number of sums and then allowing the number of sums to become infinite, he considered a weighted mean of the infinite set of sums and allowed the weights to vary in a definite manner. Thus he formed

$$F(x) = \frac{\sum_{n=0}^{\infty} s_n \dfrac{x^n}{n!}}{e^x} \tag{8}$$

and considered the limit of $F(x)$ as x becomes infinite. This definition is of interest on account of being a logical development from Cesàro's scheme, but a closely related definition which is obtained by a simple transformation of this one, is more useful in the analytic applications. We set

$$u(x) = u_0 + u_1 x + u_2 \frac{x^2}{2!} + \cdots + u_n \frac{x^n}{n!} + \cdots \tag{9}$$

and form

$$\int_0^\lambda e^{-x} u(x) dx. \tag{10}$$

If the limit of (10) as λ becomes infinite exists, we call this limit the generalized sum of the series. This definition has substantially the same scope as the earlier one.

The application of Borel's scheme to the power series development of $1/(1-z)$ is extremely simple and serves to show the scope of the method. We set

$$u(x, z) = 1 + zx + \frac{z^2 x^2}{2!} + \cdots = e^{zx}. \tag{11}$$

Then we form

$$\int_0^\lambda e^{-x} u(x, z) dx = \int_0^\lambda e^{(z-1)x} dx = \frac{e^{(z-1)\lambda}}{(z-1)} - \frac{1}{z-1}. \tag{12}$$

It is apparent that for all values of z in the complex plane such that the real part of z is less than 1, the first term on the right hand side of (12) approaches zero as λ becomes infinite. Therefore for all such values of z Borel's method serves to sum the series (7) to the appropriate value $1/(1-z)$. We have thus vastly extended the region in which the power series development can be utilized.

Shortly after Borel's investigations Leroy devised a more powerful method of summing a divergent series, which may be defined as follows. Given any series, $\Sigma u_n(z)$, we set

$$F(z, t) = \sum_{n=0}^{\infty} \frac{\Gamma(nt+1)}{\Gamma(n+1)} u_n(z) \qquad (0 < t < 1).$$

and define the value of the series as $\lim_{t \to 1-0} F(z, t)$, provided that this limit exists. It can readily be shown that Leroy's method will sum the series (7) to the value $1/(1-z)$ for all values of z in the complex plane except real values greater than one. In 1920, some twenty years after Leroy's work, Mittag-Leffler devised a method which will sum the series (7) to the value $1/(1-z)$ for all values of z for which the latter expression is defined, that is for all values of z except $z=1$.

We have seen how the various methods of summation serve to extend the region in which the series (7) can be summed to the value $1/(1-z)$. These methods are equally effective in the case of power series in general, provided that the function to which the power series corresponds is analytic in certain regions beyond the circle of convergence. They thus furnish an analytic extension of the function. To show the scope of the methods we shall state the facts in the case where the function is analytic throughout the complex plane, except for a finite number of singular points, none of which is at the origin. Suppose that we use Cauchy's generalization of Maclaurin's theorem to obtain the development $a_0+a_1z+a_2z^2+\cdots$. This series will converge within a circle whose center is at the origin and whose radius is the distance from the origin to the nearest singular point. Draw lines through the singular points perpendicular to the lines joining these points with the origin. These lines form a polygon, known as the Borel polygon, within which the series is summable to the value of the function by Borel's method. Draw lines from the singular points to infinity that are the prolongations of the lines from the origin to these singular points. The entire plane, except the lines extending out from the singular points, forms a region known as the Mittag-Leffler star. Leroy's method will sum the series to the value of the function everywhere in the Mittag-Leffler star. Finally, Mittag-Leffler's method will sum the series to the function everywhere, except at the singular points themselves.

5. *Summation of Fourier series and related developments.* We have seen what could be done with power series by the use of general methods of summation. Let us turn now to some other types of series and consider first of all the important class known as Fourier series. The Fourier series associated with a function $f(x)$, defined and integrable in the interval $(-\pi, \pi)$, may be written in the form

$$\tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + \cdots + (a_n \cos nx + b_n \sin nx) + \cdots, \tag{13}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx. \tag{14}$$

If the function satisfies certain fairly general conditions, the series will converge to the value of the function at all points of continuity of the function. However, there exist continuous functions for which the series fails to converge at points that are densely distributed in every sub-interval of the interval $(-\pi, \pi)$. Shortly after Borel's work on power series, it occurred to Fejér that some of the methods for summing divergent series might be applied to advantage in the case of Fourier series. An investigation of the matter showed that this was indeed the case and led to the following noteworthy result. *If the function of which the Fourier series is a development is finite except at a finite number of points and is integrable in the sense of Riemann, throughout the interval* $(-\pi, \pi)$, *then the Fourier series is summable* $(C1)$ *to the value of the function at all points of continu-*

ity and to the value $\frac{1}{2}[f(x+0)+f(x-0)]$ *at all points of discontinuity where this expression has a meaning.*

We have spoken earlier of developments analogous to Fourier series, such as developments in Laplace's functions or Bessel's functions, which likewise have important applications in mathematical physics. Shortly after his investigation of Fourier series Fejér made an analogous study of developments in Laplace's functions. A little later C. N. Moore investigated the developments in Bessel's functions, and Haar studied the developments in Sturm-Liouville functions. In all these cases it was the method of summation due to Cesàro that proved fruitful. In general the results were analogous to those already obtained for Fourier series, but in the case of developments in Laplace's functions and Bessel's functions certain important differences were found in the neighborhood of the point where the differential equation defining these functions has a singular point.

The investigations just described were merely the first applications of general methods of summation to the developments in question. Many subsequent studies have been made and the field of research opened up by the initial investigations is still far from exhausted. It is not possible to give any detailed account of the various results that have been obtained in a general lecture on the whole subject of summability of series. It may be stated, however, that the papers in which these results are found form an important part of the mathematical literature of the present century.

6. *Summation of Dirichlet's series.* There is one further type of series to which we have referred in the introduction, namely series of the form $\Sigma a_n e^{-\lambda_n s}$, known as Dirichlet's series. A brief account of some of the investigations of these series seems desirable, since it will show the importance of modifying various methods of summability in order to adapt them to the specific needs of certain types of series. The first investigations of the summability of Dirichlet's series were applications of Cesàro's method to certain special types of Dirichlet's series by Bohr and M. Riesz. In passing to the case of the general Dirichlet's series, Riesz found it desirable to replace Cesàro's method by a more general one, which he termed summation by typical means. For the type of series initially studied, summation by Riesz's typical mean is equivalent to summation by a corresponding Cesàro mean, but even in this case the method due to Riesz is a more effective weapon in studying the behavior of Dirichlet's series.

7. *Conclusion.* The preceding account of what has been accomplished by the use of various methods of summation in the investigation of divergent series is necessarily somewhat sketchy. I trust that it will serve to give some idea of the very notable success that has attended the application of these methods. It is easy to see that among series in general a convergent series is a very exceptional case. For convergence it is necessary first of all that the general term of the series approach zero as a limit, and many important series where the general term does approach zero fail to converge. It is apparent that if we are to achieve any comprehensive knowledge of series we must make use of methods of evaluat-

ing divergent series. The methods which we have described have amply proved their worth in the studies that have been made. We have every reason to expect that further application of these methods and others yet to be devised will vastly extend our knowledge of the important field of mathematics known as the theory of infinite series.

ON THE GEOMETRY OF THE TRIANGLE[1]

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Introduction. The purpose of Part A of this discussion is to develop an analytic treatment, using conjugate coordinates, of the extensive metric figure associated with an arbitrary triangle and to present some inter-relations in this figure which are believed to be new. Comprehensive bibliographies covering known relations can be found in *Mathesis*.[2] Part B is concerned with certain projective properties.

Part A

I. *Conjugate Coordinates.*

1. The conjugate coordinates[3] x and y of a point in a plane are defined by the equations, $x=X+iY$, $y=X-iY$, where X, Y are rectangular Cartesian coordinates and i is the imaginary unit. A complex number whose absolute value is unity is called a turn and will be denoted by such symbols as t, T, t_j, etc. The operations of multiplication, division and extraction of roots applied to turns give results which are turns.

2. A real line is represented by a linear equation $tx+y=ct$, where c is any complex number, $\bar{c}$ is the conjugate of c, and t a turn such that $\bar{c}=ct$. The point c is the reflexion of the origin in the line, and the turn t gives the inclination of the line and will be called the clinant. Two lines with clinants t_1 and t_2 are parallel if $t_1=t_2$ and perpendicular if $t_1=-t_2$. The equation of the line whose clinant is t and which passes through the point a is $tx+y=ta+\bar{a}$. The equation of the line through the points a and b is

$$\begin{vmatrix} x & y & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix} = 0.$$

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[1] Accepted for publication during the editorship of W. H. Bussey.

[2] *Mathesis Supplements*, vol. 6 (1886), vol. 10 (1890), vol. 15 (1895), vol. 27 (1907), and Year 1922 beginning page 50.

[3] W. B. Carver, *The failure of the Clifford chain*, American Journal Mathematics vol. 42 (1920), p. 137; Frank Morley, *On the metric geometry of the plane N-line*, Transactions of the American Mathematical Society, vol. 1 (1900), p. 97; R. M. Winger, *An Introduction to Projective Geometry*, D. C. Heath and Co. (1923), p. 324.

3. If the point a is the center of a circle and $|r|$ its radius, its map equation is $x=a+rt$. The conjugate equation of this circle is $(x-a)(y-\bar{a})=r\bar{r}$, where the product $r\bar{r}$ is real and greater than 0 and is the square of the radius.

II. *The Incenter and Excenters.*

1. The origin of the system of coordinates is taken at O, the circumcenter of the triangle ABC. No generality is lost by taking the circumcircle as the unit circle and assigning to the vertices A, B and C the turn values t_1, t_2 and t_3 respectively. The orthocenter of such a triangle is given by the sum of the coordinates of its vertices, that is $t_1+t_2+t_3$.

2. The points A and B divide the unit circle into two arcs AB and the midpoints of these arcs are given by $\pm(t_1t_2)^{1/2}$. We may define T_3 as that one of the square roots of t_1t_2 which represents the midpoint of that arc AB which contains the point C. A similar agreement defines T_1 and T_2. It is evident at once that $T_1T_2 \div T_3 = \pm t_3$, and it can be shown that under the above definition $T_1T_2 \div T_3 = +t_3$; similarly that $T_2T_3 \div T_1 = t_1$ and $T_3T_1 \div T_2 = t_2$.

3. It is readily seen that the orthocenter of the triangle $(-T_1)(-T_2)(-T_3)$ is the incenter I_0 of ABC and that the orthocenter of the triangle $(-T_1)(+T_2)(+T_3)$ is the excenter I_1 opposite A of ABC. The coordinates of the points $I_i (i=0 \cdots 3)$ then follow.

$$
\begin{aligned}
I_0 &= -T_1 - T_2 - T_3 = -\Sigma_1, \\
I_1 &= -T_1 + T_2 + T_3 = \Sigma_1 - 2T_1, \\
I_2 &= T_1 - T_2 + T_3 = \Sigma_1 - 2T_2, \\
I_3 &= T_1 + T_2 - T_3 = \Sigma_1 - 2T_3,
\end{aligned}
$$

where[1] $\Sigma_1 = T_1+T_2+T_3$.

If A', B', C', I_i' are the reflexions of A, B, C, $I_i (i=0 \cdots 3)$ respectively through O, then their coordinates are the negative of the coordinates of the latter. Agronomof[2] has pointed out that the circumcenters of the triangles formed from the set I_i' taken three at a time form the orthocentric set of points I_i and visa versa.

4. If the triangle ABC is isosceles, the relation among the turns T_i is $T_1^2 = T_2T_3$, $T_2^2 = T_1T_3$ or $T_3^2 = T_1T_2$ depending whether the equal sides meet in A, B, or C. Should any two of these conditions hold at the same time, the third necessarily does and the triangle is equilateral. Throughout the rest of this paper (except §XIII) it will be assumed that t_1, t_2, t_3 and T_1, T_2, T_3 do not satisfy any of these conditions, i.e., that the triangle ABC is neither isosceles or equilateral.

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[1] $\Sigma_i (i=1, 2, 3)$ will be used to denote the usual symmetric functions of the T's; i.e., the T's are the roots of the equation $T^3 - \Sigma_1 T^2 + \Sigma_2 T - \Sigma_3 = 0$.

[2] Mathesis, vol. 27 (1907), pp. 14–15.

III. *A set of Transformations.*

1. To facilitate our procedure we will define a set of transformations $J_i(i=1, 2, 3)$ which will have the meaning that the application of J_i to an expression in $T_j(j=1, 2, 3)$ will change the signs of every T except T_i. For instance, $J_1(I_0)=I_1$, $J_2(I_0)=I_2$, $J_3[J_2(I_0)]=I_1$, etc.

IV. *The Points of Contact A_i, B_i, C_i of the In- and Ex-circles (Centers at $I_i(i=0 \cdots 3)$) with the Sides BC, CA, AB Respectively.*

1. By eliminating y between $(1/T_1^2)x+y=(T_2^2+T_3^2)/\Sigma_3$ and $(-1/T_1^2)x+y=\Sigma_1/T_1^3-\bar{\Sigma}_1$, which are respectively the equations of BC and the line through I_0 perpendicular to BC, we find

$$A_0 = [T_1T_2^2 + T_1^2T_2 + T_1T_3^2 + T_1^2T_3 - T_2^2T_3 - T_2T_3^2] \div [2T_2T_3].$$

Similarly

$$B_0 = [T_1T_2^2 + T_1^2T_2 + T_2T_3^2 + T_2^2T_3 - T_1^2T_3 - T_1T_3^2] \div [2T_1T_3],$$
$$C_0 = [T_2T_3^2 + T_2^2T_3 + T_1T_3^2 + T_1^2T_3 - T_1^2T_2 - T_1T_2^2] \div [2T_1T_2].$$

The application of $J_i(i=1, 2, 3)$ to A_0, B_0, C_0 will give A_i, B_i, C_i. The triangle $A_iB_iC_i$ can not be a right triangle, as this would make two sides of the triangle ABC parallel.

V. *The Radii of the In- and Ex-circles.*

1. The radius r_0 (positive) of the incircle, center at I_0, is $[(A_0-I_0)(\bar{A}_0-\bar{I}_0)]^{1/2}$, or $(\Sigma_3-\Sigma_1\Sigma_2)/2\Sigma_3$. The application of $J_i(i=1, 2, 3)$ to $-r_0$, that is, to $(\Sigma_1\Sigma_2-\Sigma_3)/2\Sigma_3$, will give r_i (also positive).

VI. *The Symmedian Points of the Triangles $A_iB_iC_i(i=0 \cdots 3)$.*

1. In an arbitrary triangle XYZ whose vertices are the turn values u_1, u_2, u_3 respectively, the symmedian point is

$$(a) \qquad [6F_1F_3 - 2F_2^2]/[9F_3 - F_1F_2],$$

where $F_i(i=1, 2, 3)$ denote the ordinary symmetric functions of u_1, u_2, u_3. It can be shown that if u_1, u_2, u_3 are turns, $9F_3-F_1F_2$ can vanish only if $u_1=u_2=u_3$, hence the expression (a) always has a meaning.

2. To find the symmedian point K_0 of the triangle $A_0B_0C_0$ the coordinate system may be shifted by means of the transformation $x'=(x-I_0)/r_0$. This transformation shifts the origin to I_0 and causes a stretching (or shrinking) so that the incircle whose radius is r_0 becomes the unit circle, thereby making A_0, B_0, C_0 have turn values. If the new coordinates of the points A_0, B_0, C_0 are denoted by A_0'', B_0'', C_0'' respectively, then

$$A_0'' = 2\Sigma_3(A_0 - I_0)/(\Sigma_3 - \Sigma_1\Sigma_2) = -T_1.$$

Simiarly $B_0''=-T_2$ and $C_0''=-T_3$. $\Sigma_1\Sigma_2-\Sigma_3=(T_2+T_3)(T_3+T_1)(T_1+T_2)$. If $T_2+T_3=0$, $\sqrt{t_3t_1}=\pm\sqrt{t_1t_2}$, or $t_3=t_2$. Hence $\Sigma_1\Sigma_2-\Sigma_3$ can not vanish if the vertices of ABC are distinct. Making use of (a),

$$K_0'' = (2\Sigma_2^3 - 6\Sigma_1\Sigma_3)/(9\Sigma_3 - \Sigma_1\Sigma_2).$$

The inverse transformation $x = r_0x' + I_0$ applied to this point will give its coordinate with respect to the original system of coordinates. By this means the symmedian point K_0 of $A_0B_0C_0$ is found to be

$$[4\Sigma_1^3\Sigma_2\Sigma_3 - \Sigma_1\Sigma_2^3 + \Sigma_2^3\Sigma_3 - 12\Sigma_1\Sigma_3^2]/\Sigma_3[9\Sigma_3 - \Sigma_1\Sigma_2].$$

The application of $J_i (i = 1, 2, 3)$ to this expression will give K_i, the symmedian point of the triangle $A_iB_iC_i$.

VII. *A Collinear Relation of Three Points.*

1. Theorem A: *The nine point center N and the symmedian point K of a triangle XYZ (which is not a right triangle) are collinear with the orthocenter H of the tangential triangle $\alpha\beta\gamma$ of XYZ.*

2. Again taking for the vertices XYZ the turn values u_1, u_2, u_3 respectively we have the following tabulation:

$$\alpha = 2u_2u_3/(u_2 + u_3), \quad \beta = 2u_1u_3/(u_1 + u_3), \quad \gamma = 2u_1u_2/(u_1 + u_2),$$
$$H = (2F_2^3 - 2F_1F_3)/(F_1F_2 - F_3).$$

$(F_1F_2 - F_3 \neq 0$ if XYZ is not a right triangle). Since the nine point center is midway between the orthocenter and the circumcenter we have $N = F_1/2$; and, from §VI-1, $K = (6F_1F_3 - 2F_2^2)/(9F_3 - F_1F_2)$. Since

$$\begin{vmatrix} H & \overline{H} & 1 \\ N & \overline{N} & 1 \\ K & \overline{K} & 1 \end{vmatrix}$$

is found to vanish, it follows that H, N, and K are collinear.

3. In the general figure the triangle ABC is the common tangential triangle of the triangles $A_iB_iC_i (i = 0, 1, 2, 3)$. If the orthocenter of ABC is Q, then as a consequence of Theorem A there exists on Q a pencil of four lines N_iK_iQ, where N_i is the nine point center of the triangle $A_iB_iC_i$.

VIII. *The Hessian Lines of the Figure.*

1. Theorem B: *It is known that in any triangle XYZ[1] the circumcenter O and the symmedian point K lie on the Hessian line[2] of the triangle. The orthocenter P of the pedal triangle and the reflexion H' of the orthocenter of the tangential triangle through its circumcenter are also on the Hessian line and satisfy the harmonic relation $[OPKH']$.*[3]

[1] To meet later needs of the theorem, XYZ must not be a right triangle.

[2] This line is so named because the Hessian (equated to 0) of the cubic whose roots are the turns representing the vertices of XYZ defines two so called Hessian points which with O and K satisfy a harmonic relation upon the line. The linear relation OKP was known to Tucker; see p. 175 of Milne's *Companion to the Weekly Problem Papers* (Macmillan and Co., 1888).

[3] Throughout this discussion $[ABCD]$ will represent a harmonic range of points where C and D are harmonic conjugates with respect to A and B.

2. Using u_1, u_2, u_3 as the coordinates of the vertices of XYZ, the coordinates of the points concerned are:
$O=0$, $K=(6F_1F_3-2F_2^2)/(9F_3-F_1F_2)$, $P=(3F_1F_3-F_2^2)/2F_3$, $H'=(6F_1F_3-2F_2^2)/(F_1F_2-F_3)$. The equation of OK is

$$(F_1^2 - 3F_2)x + (3F_1F_3 - F_2^2)y = 0.$$

It may be easily verified that P and H' satisfy this equation, thereby showing that O, K, P and H' are collinear points. In order to show that these points form a harmonic set $[OPKH']$ it is necessary that

$$[(O - K)(P - H')]/[(O - H')(P - K)] = -1,$$

or, since O is at the origin,

$$P = 2KH'/(K + H'),$$

and a direct substitution from above will substantiate this harmonic property.

3. Now consider any one of the four triangles (not right triangles) $A_iB_iC_i (i=0, 1, 2, 3)$ (or any one of the four triangles $A_i'B_i'C_i'$). Its circumcenter is at $I_i(I_i')$, the orthocenter of its pedal triangle is $P_i(P_i')$, and the orthocenter of its tangential triangle reflected through O is $Q'(Q)$. Therefore there are four Hessian lines concurrent at Q' and four at Q, each bearing a harmonic set of points $[I_iP_iK_iQ'](I_i'P_i'K_i'Q])$. The former come from the triangles $A_iB_iC_i$ and the latter from the triangles $A_i'B_i'C_i'$.

4. Each of the eight Hessian lines of the triangles $A_iB_iC_i$ and $A_i'B_i'C_i'$ is also the Hessian line of another triangle. The Hessian line $I_iQ'(I_i'Q)$ of the triangle $A_iB_iC_i(A_i'B_i'C_i')$ is the Hessian line of the triangle whose vertices are those points of the set $I'(I)$ which bear subscripts different from i.

IX. *A Relation Between Certain Symmedian Points.*

1. Theorem C: *The symmedian point L_i' $(i=0, 1, 2, 3)$ (L_i) of a triangle whose vertices are those points of the set $I'(I)$ which bear subscripts different from i is mid-way between the orthocenter Q' of $A'B'C'$ $(Q$ of $ABC)$ and the symmedian point $K_i(K_i')$ of the triangle $A_iB_iC_i(A_i'B_i'C_i')$.*

2. It will be shown first that L_0', the symmedian point of the triangle $I_1'I_2'I_3'$ (circumcenter at I_0) is mid-way between Q' and K_0. From §II-3,

$$I_0 = -\Sigma_1, \quad I_1' = -\Sigma_1 + 2T_1, \quad I_2' = -\Sigma_1 + 2T_2, \quad \text{and} \quad I_3' = -\Sigma_1 + 2T_3.$$

Then by means of the transformation $x'=(x+\Sigma_1)/2$ and its inverse we are able to find

$$L_0' = [3\Sigma_1\Sigma_3 - 4\Sigma_2^2 + \Sigma_1^2\Sigma_2]/[9\Sigma_3 - \Sigma_1\Sigma_2].$$

The points Q' and K_0 are respectively

$$[2\Sigma_1\Sigma_3 - \Sigma_2^2]/\Sigma_3 \text{ and } [4\Sigma_1^2\Sigma_2\Sigma_3 - \Sigma_1\Sigma_2^3 + \Sigma_2^2\Sigma_3 - 12\Sigma_1\Sigma_3^2]/\Sigma_3[9\Sigma_3 - \Sigma_1\Sigma_2] (\S VI\text{-}2).$$

Since $Q'+K_0=2L_0'$, the fact that L_0' is mid-way between Q' and K_0 is established.

3. The application of the transformations $J_i(i=1, 2, 3)$ to the numbers throughout the above work will establish this property for the other Hessian lines concurrent at Q'. From the symmetry of the figure it is clear that whatever is true of the pencil of Hessian lines on Q' is also true of the pencil on Q.

X. *The Four Feuerbach Points.*[1]

1. If $\phi_i(i=0, 1, 2, 3)$ represents the point of contact of the in- or ex-circle (center at I_i) and the nine point circle of ABC, we have

$$\phi_0 = [\Sigma_2^2 - 2\Sigma_1\Sigma_3]/2\Sigma_3 - \Sigma_2/2\Sigma_1.$$

$\Sigma_1 \neq 0$, for if $\Sigma_1=0$ the triangle ABC must be isosceles. The application of $J_i(i=1, 2, 3)$ to ϕ_0 gives ϕ_i. These points $\phi_i(i=1, 2, 3)$ always exist, since the denominators involved can never vanish.

XI. *The Four Points* $\lambda_i(i=0, 1, 2, 3)$ *of Lemoine.*

1. The point λ_i is defined[2] as the harmonic conjugate of ϕ_i in the set $[\phi_i\lambda_i I_i R]$, where R is the center of the nine point circle containing the ϕ's. The point λ_i will exist except in the case where ϕ_i is the mid-point of I_iR. It is clear that ϕ_i, I_i and R are collinear from their definitions. λ_0 can be found by substituting I_0 (§II-3), ϕ_0 (§X-1) and $R=[\Sigma_2^2-2\Sigma_1\Sigma_3]/2\Sigma_3$ into the expression

$$[(\phi_0 - I_0)(\lambda_0 - R)]/[(\phi_0 - R)(\lambda_0 - I_0)] = -1,$$

thereby giving

$$\lambda_0 = [\Sigma_2^2 - 2\Sigma_1\Sigma_3]/2\Sigma_3 + \Sigma_2^2/2[\Sigma_1\Sigma_2 - 2\Sigma_3].$$

The denominator

$$\Sigma_1\Sigma_2 - 2\Sigma_3 = \Sigma_3(\Sigma_1\Sigma_2/\Sigma_3 - 2) \neq 0$$

in as much as $\Sigma_1\Sigma_2/\Sigma_3$ (the square of the distance OI_0) is always less than unity. In other words, ϕ_0 is never the mid-point of the segment I_0R. The application of $J_i(i=1, 2, 3)$ to λ_0 gives λ_i. The application of J_1 to $(\Sigma_1\Sigma_2-2\Sigma_3)/2\Sigma_3$ gives

$$\left[\frac{(T_1 - T_2 - T_3)(-T_1T_2 - T_1T_3 + T_2T_3) - \Sigma_3}{2T_1T_2T_3}\right] - \tfrac{1}{2}.$$

However, the expression in brackets is r_1 (§V-1) and, hence, the denominator of λ_1 vanishes if $r_1=1/2$. In general, then, if the radius r_i of the circumcircle of $A_iB_iC_i(i=1, 2, 3)$ equals $1/2$, ϕ_i bisects the segment I_iR and the point λ_i does not exist.

[1] Mathesis, vol. 36, pp. 50–51.

[2] This is not Lemoine's definition of these points. He defines them by the linear properties in XI–2. See Nouvelles Annales de Mathématiques, 3 série, tome 5 (1886), p. 122.

2. Historically (see last footnote), these λ's are not defined from the above harmonic property, but rather, $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are defined as the respective points of concurrence of the following groups of lines:

$$\begin{array}{llll} A\phi_1 & A\phi_0 & A\phi_3 & A\phi_2 \\ B\phi_2, & B\phi_3, & B\phi_0, & A\phi_1. \\ C\phi_3 & C\phi_2 & C\phi_1 & C\phi_0 \end{array}$$

XII. *A Four Line* $p_i (i=0, 1, 2, 3)$.

1. It is obvious that the line joining I_i and I_j, where $i \neq j$, and $i, j = 0, 1, 2, 3$, passes through the corresponding[1] vertex of the triangle ABC. The point P_{ij} will be defined as the harmonic conjugate of this vertex with respect to I_i and I_j. It is well known[2] that these six points are the vertices of a four-line having the same diagonal triangle ABC as the four-point $I_i (i=0, 1, 2, 3)$. The lines of the four-line will be represented by p_i $(i=0, 1, 2, 3)$, where P_{ij} is not the intersection of p_i and p_j, but the intersection of the other two lines of the four-line. It will be shown that the four lines are the axes of perspectivity of the pairs of triangles with centers of perspectivity at the ϕ's and λ's (§XI-2); and also that each of the six points P_{ij} lies on the corresponding line of each of the complete four-points made up of the ϕ's, the I's and the λ's; i.e. P_{ij} is the point of concurrence of the lines $\phi_i\phi_j$, I_iI_j and $\lambda_i\lambda_j$.

2. From the definitions $[I_0I_1P_{01}A]$, $[I_0I_2P_{02}B]$, $[I_0I_3P_{03}C]$ we obtain, respectively,

$$\begin{aligned} P_{01} &= T_1(T_2^2 + T_3^2 - T_1^2 + T_2T_3)/(T_1^2 + T_2T_3), \\ P_{02} &= T_2(T_1^2 + T_3^2 - T_2^2 + T_1T_3)/(T_2^2 + T_1T_3), \\ P_{03} &= T_3(T_2^2 + T_1^2 - T_3^2 + T_1T_2)/(T_3^2 + T_1T_2). \end{aligned}$$

The points P_{12}, P_{23}, P_{31} could be obtained likewise from their definitions $[I_1I_2P_{12}C]$, $[I_3I_2P_{23}A]$, $[I_1I_3P_{31}B]$ respectively, but we note in a simpler manner that $P_{12} = J_1(P_{03}) = J_2(P_{03})$, $P_{23} = J_2(P_{01}) = J_3(P_{01})$, $P_{31} = J_1(P_{02}) = J_3(P_{02})$. The expressions for P_{01}, P_{02} and P_{03} always have a meaning and those for P_{12}, P_{23} and P_{31} do, provided the original triangle is not isosceles (§II-4).

3. By direct substitution it may now be shown that each of these points P_{ij} lies on three lines, as follows:

$$\begin{array}{ll} P_{01} \text{ on } BC,\ \phi_0\phi_1,\ \lambda_0\lambda_1, & P_{12} \text{ on } AB,\ \phi_1\phi_2,\ \lambda_1\lambda_2, \\ P_{02} \text{ on } AC,\ \phi_0\phi_2,\ \lambda_0\lambda_2, & P_{13} \text{ on } AC,\ \phi_1\phi_3,\ \lambda_1\lambda_3, \\ P_{03} \text{ on } AB,\ \phi_0\phi_3,\ \lambda_0\lambda_3, & P_{23} \text{ on } BC,\ \phi_2\phi_3,\ \lambda_2\lambda_3. \end{array}$$

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[1] This correspondence is between the letters ABC and the three ways of separating 0123 into two pairs, thus, A with 01–23, B with 02–13, etc.

[2] A. Clebsch, *Leçons sur la Géométrie*, tome 1, p. 71.

These linear relations when considered in connection with the pencils of lines on the ϕ's and λ's (XI-2) establish the fact that the points P_{ij} are arranged by threes upon the four lines $p_i (i=0, 1, 2, 3)$ which are the axes of perspectivity of the triangles perspective from the ϕ's and from the λ's. The significance of the subscripts is that p_i is that line of the four-line which contains the three of the six points P_{ij} which do not bear the subscript i.

4. In going back to the definition (XI-1) of the points λ_i it will be recalled that there exists on R a pencil of four lines of the type $RI_i\phi_i\lambda_i (i=0, 1, 2, 3)$. If all the possible pairs of triangles perspective from R are selected (where each triangle will have vertices consisting entirely of I's, ϕ's or λ's), then in the light of the theorem of Desargues there will be certain axes of perspectivity. Since the anharmonic ratio of the points on each line is constant for the whole pencil, there are not twelve axes of perspectivity, but only four, and these constitute the four-line p_i.

5. By selecting proper four-points or by direct substitution it can be shown that these points P_{ij} enter into other harmonic relations: $[CAP_{02}P_{13}]$, $[BCP_{01}P_{23}]$ and $[ABP_{03}P_{12}]$.

XIII. *Some Special Cases.*

1. Thus far in this discussion there have been three types of triangles ABC which have caused vanishing denominators: (1) isosceles triangles, (2) equilateral triangles, and (3) triangles having a radius of an escribed circle equal to 1/2. Type (3) will receive no further treatment than that already given it (XI). Types (1) and (2), however, have been ruled out of the past discussion, and more consideration will be given them now.

2. *The isosceles triangle.* The triangle ABC is isosceles under any one of three conditions: $T_1^2 = T_2T_3$, $T_2^2 = T_1T_3$, or $T_3^2 = T_1T_2$, depending on whether its equal sides meet in A, B, or C. It will suit the present purpose to investigate the figure when ABC is isosceles due to the condition $T_1^2 = T_2T_3$, with $T_2^2 \neq T_1T_3$ and $T_3^2 \neq T_1T_2$. For such a triangle all the expressions representing points for the general case will apply to this case when these expressions do not contain vanishing denominators. Thus, all the points I_i, ϕ_i, λ_i, P_{ij} (except P_{23}), K and R exist and are represented by the same expressions which represent them in the general case. The nine point circle of ABC and the two in- and ex-circles with centers at I_0 and I_1 are tangent to each other at the mid-point of BC; therefore, $\phi_0 = \phi_1 = (t_2+t_3)/2$. In satisfying the harmonic relation $[I_0I_1P_{01}A]$, P_{01} becomes $(t_2+t_3)/2$ also. The segment I_2I_3 is bisected by A, therefore there is no point P_{23}. The remaining P_{ij} points exist and the four-line p_i is special only in the sense that p_0 and p_1 are parallel. The points λ_2 and λ_3 are on BC. The perpendicular bisector of BC contains the points A, O, I_0, λ_0, R, K, $\phi_0 = \phi_1 = P_{01}$ and I_1 and this line is both the Euler and the Hessian line of ABC.

3. *The equilateral triangle.* The conditions for an equilateral triangle ABC are $T_1^2 = T_2T_3$, $T_2^2 = T_1T_3$ and $T_3^2 = T_1T_2$. In order to simplify the discussion let the axis of reals be located so that $T_1 = t_1 = 1$. Then $T_2T_3 = 1$, $T_2^2 = T_3$, $T_3^2 = T_2$,

from which it follows that $T_2=\omega$, $T_3=\omega^2$, or $T_2=\omega^2$, $T_3=\omega$, where $1, \omega, \omega^2$ represent the cube roots of 1. The expressions for the points I_i obtained in §II evidently hold for the equilateral triangle and we have $I_0=-(1+\omega+\omega^2)=0$, $I_1=-2$, $I_2=-2\omega$, and $I_3=-2\omega^2$. Since the nine point circle of ABC is the same as the incircle with center at I_0, the Feuerbach point ϕ_0 is indeterminate. However, $\phi_1=-1/2$, $\phi_2=-\omega/2$ and $\phi_3=-\omega^2/2$, which are the same as the results obtained by substituting in the expressions for the ϕ-points in the general case. Similarly for the λ's; λ_0 does not exist and $\lambda_1=1=A$, $\lambda_2=\omega=B$, and $\lambda_3=\omega^2=C$. The points P_{12}, P_{13}, P_{23} and the line p_0 do not exist, in as much as C, B and A bisect the line segments I_1I_2, I_1I_3 and I_2I_3 respectively. We also find that $P_{01}=-1/2=\phi_1$, $P_{02}=-\omega/2=\phi_2$, and $P_{03}=-\omega^2/2=\phi_3$. The Euler and Hessian lines are both indeterminate.

Part B

XIV. *Projective Properties.*

1. Except for certain classes of triangles which have been specially treated (XIII), the points and lines associated with the triangle have been the points and lines of ordinary geometry and not the points and lines at infinity of projective geometry. From now on no distinction will be made and throughout this section the words point and line will be used in the broader sense of projective geometry.

XV. *The Set of Points Q_{ij}.*

1. There are six points determined by the six pairs of lines $(\phi_i\phi_j)$ and $(\lambda_k\lambda_m)$, and these points constitute the set Q_{ij}, where i, j, k, m are distinct and equal 0, 1, 2, 3. For example, Q_{01} is the intersection of the lines $\phi_0\phi_1$ and $\lambda_2\lambda_3$.

2. From XI-2 there exists a pencil of four lines on each of the points A, B, C. The pencil on A is

$$\begin{matrix}\lambda_2\phi_3 & \lambda_1\phi_0 \\ \lambda_1\phi_0 & \lambda_3\phi_2 \\ \phi_1\lambda_0 & \phi_1\lambda_0 \\ \phi_2\lambda_3 & \phi_3\lambda_2\end{matrix} \quad \text{or} \quad .$$

With A as center of perspectivity there are four pairs of perspective triangles considering all the possible triangles from each column as arranged above. The corresponding sides define the harmonic sets of points $[RCQ_{03}Q_{12}]$ and $[RBQ_{02}Q_{13}]$. Similarly, using B and C as centers we get respectively $[RAQ_{01}Q_{23}]$, $[RCQ_{03}Q_{12}]$ and $[RAQ_{01}Q_{23}]$, $[RBQ_{02}Q_{13}]$.

3. There exist twelve lines of the type $Q_{ij}Q_{jk}P_{ik}$, where i, j, k are distinct and equal 0, 1, 2, 3. Consider the triangles $\phi_0\phi_2\phi_1$ and $\lambda_2\lambda_0\lambda_3$ perspective from B. The axis of perspectivity is $Q_{01}Q_{12}P_{02}$. The complete truth of the statement may be verified by selecting other perspective triangles.

4. Then there exist twelve harmonic sets of points, six of the type

$[P_{ij}Q_{ij}\phi_i\phi_j]$ and six of the type $[P_{ij}Q_{km}\lambda_i\lambda_j]$, where i, j, k, m are distinct and equal to 0, 1, 2, 3. These may be arrived at by the proper selections of complete four-points.

XVI. *The Complete Pencil of Lines on R.*

1. On the seven lines through R there exist the following seven harmonic sets, the last three being repeated with the Q's interchanged (XI-1, XV-2):

$$RI_0\phi_0\lambda_0,\ RI_2\phi_2\lambda_2,\ RAQ_{01}Q_{23},\ RBQ_{02}Q_{13},\ RCQ_{03}Q_{12},$$
$$RI_1\phi_1\lambda_1,\ RI_3\phi_3\lambda_3,\ RAQ_{23}Q_{01},\ RBQ_{13}Q_{02},\ RCQ_{12}Q_{03}.$$

If these ten sets were arranged in a single column and if all the possible pairs of perspective triangles were selected by using as vertices of one triangle three points selected from one column within the tabulation and for the other triangle the corresponding three points selected from either other column, the axis of perspectivity would be, in each case, one of the four lines p_i (XII). The point of intersection of any corresponding sides will be either a point P_{ij} or one of twelve new points which will be denoted by T_{i-jk}, where i, j, k are distinct and equal to 0, 1, 2, 3. In designating the T's as the points of concurrence of certain sets of lines, the following notation will be clear:

$$T_{0-12} = (\phi_3Q_{12})(\lambda_3Q_{03})(I_3C), \quad T_{1-02} = (\phi_3Q_{02})(\lambda_3Q_{13})(I_3B),$$
$$T_{0-23} = (\phi_1Q_{23})(\lambda_1Q_{01})(I_1A), \quad T_{1-03} = (\phi_2Q_{03})(\lambda_2Q_{12})(I_2C),$$
$$T_{0-13} = (\phi_2Q_{13})(\lambda_2Q_{02})(I_2B), \quad T_{1-23} = (\phi_0Q_{23})(\lambda_0Q_{01})(I_0A),$$
$$T_{2-01} = (\phi_3Q_{01})(\lambda_3Q_{23})(I_3A), \quad T_{3-01} = (\phi_2Q_{01})(\lambda_2Q_{23})(I_2A),$$
$$T_{2-03} = (\phi_1Q_{03})(\lambda_1Q_{12})(I_1C), \quad T_{3-02} = (\phi_1Q_{02})(\lambda_1Q_{13})(I_1B),$$
$$T_{2-13} = (\phi_0Q_{13})(\lambda_0Q_{02})(I_0B), \quad T_{3-12} = (\phi_0Q_{12})(\lambda_0Q_{03})(I_0C).$$

2. It may be easily shown that p_i contains three points of the type T_{i-jk} in addition to the three points P_{jk}. For instance, p_0 contains P_{23}, P_{13}, P_{12}, T_{0-23}, T_{0-13} and T_{0-12}. Moreover, if P_{jk}, P_{kl} and P_{jl} are on p_i, then T_{i-jk} will be the conjugate of P_{jk} in the harmonic set $[P_{jk}T_{i-jk}P_{kl}P_{jl}]$, similarly, T_{i-kl} will be the conjugate of P_{kl}, etc. This means that if a homogeneous one-dimensional coordinate system were established on the line p_i and if the three points P_{jk}, P_{kl}, P_{jl} were given by a binary cubic form, then the three points T_{i-jk}, T_{i-kl}, T_{i-jl} would be given by the cubic covariant form.[1]

3. The T's also enter into harmonic sets of points upon the sides of the orthocentric quadrilateral (vertices at $I_i (i=0, 1, 2, 3)$). Consider the following six relations:

$$T_{1-23}T_{0-23}P_{01}AI_0I_1, \quad T_{2-03}T_{1-03}P_{12}CI_1I_2,$$
$$T_{2-13}T_{0-13}P_{02}BI_0I_2, \quad T_{3-02}T_{1-02}P_{13}BI_1I_3,$$
$$T_{3-12}T_{0-12}P_{03}CI_0I_3, \quad T_{3-01}T_{2-01}P_{23}AI_2I_3.$$

[1] A. Clebsch, *Leçons sur la Géométrie*, tome I, page 272.

On the line I_0I_1, for example, there exist the harmonic sets $[AP_{01}T_{0-23}T_{1-23}]$, $[AI_0P_{01}T_{1-23}]$, $[AI_1T_{0-23}P_{01}]$, and $[AP_{01}I_0I_1]$. (See XII-2). Similar relations can be found on the remaining lines.

XVII. *Projective Coordinates.*

1. A homogeneous coordinate system will be entirely determined if the points A, B, C and I_0 are assigned the following coordinates: $A=(1:0:0)$, $B=(0:1:0)$, $C=(0:0:1)$, and $I_0=(1:1:1)$. The P's and the remaining I's follow readily from their properties in XII. We have

$$P_{01}=(0:1:1),\quad P_{12}=(1:-1:0),\quad I_1=(-1:1:1),$$
$$P_{02}=(1:0:1),\quad P_{13}=(1:0:-1),\quad I_2=(1:-1:1),\text{ and}$$
$$P_{03}=(1:1:0),\quad P_{23}=(0:1:-1),\quad I_3=(1:1:-1).$$

Let $(a:b:c)$ be the projective coordinates of R. The point ϕ_0 lying on the line joining R and I_0 will have the projective coordinates $(va+1:vb+1:vc+1)$. We may assume that this constant v has been incorporated into the a, b, and c (which are, therefore, no longer homogeneous), and the coordinates of ϕ_0 then become $(a+1:b+1:c+1)$. Then the conditions (XV) on ϕ_i and λ_i determine their coordinates as follows:

$$\phi_0=(a+1:b+1:c+1),\qquad \lambda_0=(a-1:b-1:c-1),$$
$$\phi_1=(a+1:b-1:c-1),\qquad \lambda_1=(a-1:b+1:c+1),$$
$$\phi_2=(a-1:b+1:c-1),\qquad \lambda_2=(a+1:b-1:c+1),$$
$$\phi_3=(a-1:b-1:c+1).\qquad \lambda_3=(a+1:b+1:c-1).$$

The Q's and the T's follow directly from their definitions (XV and XVI).

$$Q_{01}=(a+1:b:c),\quad T_{0-12}=(1:1:-2),\quad T_{2-01}=(2:1:-1),$$
$$Q_{02}=(a:b+1:c),\quad T_{0-23}=(-2:1:1),\quad T_{2-03}=(-1:1:2),$$
$$Q_{03}=(a:b:c+1),\quad T_{0-13}=(1:-2:1),\quad T_{2-13}=(1:2:1),$$
$$Q_{12}=(a:b:c-1),\quad T_{1-02}=(1:2:-1),\quad T_{3-01}=(2:-1:1),$$
$$Q_{13}=(a:b-1:c),\quad T_{1-03}=(1:-1:2),\quad T_{3-02}=(-1:2:1),$$
$$Q_{23}=(a-1:b:c),\quad T_{1-23}=(2:1:1),\quad T_{3-12}=(1:1:2).$$

XVIII. *The Transformation from Conjugate to Projective Coordinates.*

1. The transformation

$$\begin{aligned}\rho x_1&=A_1X+B_1Y+C_1,\\ \rho x_2&=A_2X+B_2Y+C_2,\\ \rho x_3&=A_3X+B_3Y+C_3,\end{aligned}\tag{1}$$

where the coefficients are real and the determinant of the transformation is different from zero, connects Cartesian coordinates (X, Y) with projective co-

ordinates $(x_1:x_2:x_3)$. Conjugate coordinates (x, y) are related to the Cartesian coordinates by the equations $x=X+iY$, $y=X-iY$, or $X=(x+y)/2$, $Y=(x-iy)/2i$, and these equations substituted in (1) give a transformation of the same non-singular type.

2. In conjugate coordinates the equations of the sides BC, AC, and AB of the original triangle are respectively

$$x + T_1^2 y - T_1^2(T_2^2 + T_3^2)/\Sigma_3 = 0,$$
$$x + T_2^2 y - T_2^2(T_1^2 + T_3^2)/\Sigma_3 = 0,$$
$$x + T_3^2 y - T_3^2(T_2^2 + T_1^2)/\Sigma_3 = 0.$$

If A, B, C and I_0 have the coordinates as in XVII, the equations of BC, AC and AB are respectively $x_1=0$, $x_2=0$ and $x_3=0$. Hence, the transformation connecting the two coordinate systems will be

$$\begin{aligned} x_1 &= k_1[x + T_1^2 y - T_1^2(T_2^2 + T_3^2)/\Sigma_3], \\ x_2 &= k_2[x + T_2^2 y - T_2^2(T_1^2 + T_3^2)/\Sigma_3], \\ x_3 &= k_3[x + T_3^2 y - T_3^2(T_1^2 + T_2^2)/\Sigma_3], \end{aligned} \tag{2}$$

where k_1, k_2 and k_3 are still to be determined. This may be done by means of the point I_0 which has the projective coordinates $(1:1:1)$ and conjugate coordinates $(-\Sigma_1, -\overline{\Sigma}_1)$. Substituting in transformation (2) we get $k_1:k_2:k_3 = 1/T_1:1/T_2:1/T_3$. Therefore the required transformation is

$$\begin{aligned} x_1 &= T_2T_3x + T_1\Sigma_3 y - T_1(T_2^2 + T_3^2), \\ x_2 &= T_1T_3x + T_2\Sigma_3 y - T_2(T_1^2 + T_3^2), \\ x_3 &= T_1T_2x + T_3\Sigma_3 y - T_3(T_1^2 + T_2^2). \end{aligned} \tag{3}$$

3. If this transformation (3) is applied to the conjugate coordinates of R (see XI), its projective coordinates are found to be

$$(a:b:c) = [(T_1^4 + T_2^2T_3^2)/T_1:(T_2^4 + T_1^2T_3^2)/T_2:(T_3^4 + T_1^2T_2^2)/T_3].$$

Therefore,

$$a = k(T_1^4 + T_2^2T_3^2)/T_1,\ b = k(T_2^4 + T_1^2T_3^2)/T_2, \text{ and } c = k(T_3^4 + T_1^2T_2^2)/T_3,$$

where k is a factor of proportionality. Using the point ϕ_0, it may be shown that $k=-1/2\Sigma_3$. Whence,

$$\begin{aligned} T_1^4 + T_2^2T_3^2 + 2T_1^2T_2T_3a &= 0, \quad (\text{I}) \\ T_2^4 + T_1^2T_3^2 + 2T_1T_2^2T_3b &= 0, \quad (\text{II}) \\ T_3^4 + T_1^2T_2^2 + 2T_1T_2T_3^2c &= 0. \quad (\text{III}) \end{aligned} \tag{4}$$

If a, b, and c are given arbitrary values, we would have a projective figure which would possess all the projective properties of the metric figure. Moreover, for an arbitrary selection of a, b, and c the figure obtained would not in general be a projection of our metric figure. In order that a figure with projective co-

ordinates as given in XVII may be a projection of our metric figure, there are certain conditions imposed on a, b, and c.

4. These restriction are implied in the set of equations (4) above. If equation (II) is considered as a quadratic equation in T_1, the roots are

$$[-b \pm \sqrt{(b^2-1)}]T_2^2/T_3.$$

Substituting these values in (I) and (III) gives

$$(\alpha)\quad (-b \pm \sqrt{[b^2-1]})^4(T_2^6/T_3^6) + 2a(-b \pm \sqrt{[b^2-1]})^2(T_2^3/T_3^3) = -1,$$

$$(\beta)\quad (-b \pm \sqrt{[b^2-1]})^2(T_2^6/T_3^6) + 2c(-b \pm \sqrt{[b^2-1]})(T_2^3/T_3^3) = -1,$$

where the sign before the radical is taken the same in each equation. Either of these equations will define the ratio T_2/T_3, yet T_2/T_3 must be a turn and a restriction on b can be obtained by using that fact. Take an equation as $Ax^2+Bx+C=0$, where A, B, and C may be real or complex and x a turn. Then $\bar{C}x^2+\bar{B}x+\bar{A}=0$. Therefore, $A=k\bar{C}$, $B=k\bar{B}$ and $C=k\bar{A}$, where k is a factor of proportionality. The application of these conditions to equation (β), bearing in mind that a, b, and c are real, makes it necessary that $|b| \leqq 1$. Similarly it is necessary that $|a| \leqq 1$ and $|c| \leqq 1$. Whenever such an expression occurs as $\pm\sqrt{[a^2-1]}$, where $|a| \leqq 1$, it will be written $\pm i\sqrt{[1-a^2]}$, where $\sqrt{[1-a^2]}$ means the positive (or zero) root.

5. There are, however, further conditions on a, b, c. Let equations (α) and (β) be solved for T_2^3/T_3^3. We get, respectively,

$$\frac{-a \pm i\sqrt{[1-a^2]}}{[-b \pm i\sqrt{(1-b^2)}]^2} \text{ and } \frac{-c \pm i\sqrt{[1-c^2]}}{-b \pm i\sqrt{(1-b^2)}},$$

where the signs in the two denominators are to be the same and the signs of the numerators are independent of those in the denominators and of each other. By equating equals and simplifying, we obtain the symmetrical relation

$$(A)\qquad [-a \pm i\sqrt{(1-a^2)}][-b \pm i\sqrt{(1-b^2)}][-c \pm i\sqrt{(1-c^2)}] = 1,$$

where the three double signs are quite independent. Hence, if a, b, and c are real numbers which are each of absolute value less than or equal to 1, a necessary condition that T's can be found satisfying condition (4) is that (A) should hold for some choice of signs. It is clear that if the condition holds for some choice of signs it will hold for the opposite choice.

It may now be shown that this is a sufficient condition on a, b, and c. Given then, that $|a| \leqq 1$, $|b| \leqq 1$, and $|c| \leqq 1$, and $(-a+iA)(-b+iB)(-c+iC)=1$, where A, B, and C denote respectively $\pm\sqrt{[1-a^2]}$, $\pm\sqrt{[1-b^2]}$, and $\pm\sqrt{[1-c^2]}$ for some fixed choice of signs, values may be found for T_1, T_2, and T_3 satisfying conditions (4) as follows: Take

$$T_1^3 : T_2^3 : T_3^3 = 1/(-a+iA) : 1/(-b+iB) : 1/(-c+iC),$$

or

$$T_1^3 = K^3/(-a+iA),\quad T_2^3 = K^3/(-b+iB),\quad T_3^3 = K^3/(-c+iC),$$

where K is an arbitrary turn. Then $T_1=K/\sqrt[3]{[-a+iA]}$, $T_2=K/\sqrt[3]{[-b+iB]}$, and $T_3=K/\sqrt[3]{[-c+iC]}$. Let the cube roots in the denominators be so chosen that their product is 1. Their product for any choice will be $1, \omega$, or ω^2, but after any two of them have been fixed arbitrarily the third can be so chosen that their product is 1. We then have $T_1T_2T_3=K^3$. That the set of T's so chosen satisfy conditions (4) may be seen by direct substitution. Therefore there are nine sets of T's which satisfy the conditions. If $T_1T_2T_3$ are one such set, the nine sets are

$$\begin{array}{lll} T_1 & T_2 & T_3 \\ \omega T_1 & \omega T_2 & \omega T_3 \\ \omega^2 T_1 & \omega^2 T_2 & \omega^2 T_3 \end{array} \qquad \begin{array}{lll} T_1 & \omega T_2 & \omega^2 T_3 \\ \omega T_1 & \omega^2 T_2 & T_3 \\ \omega^2 T_1 & T_2 & \omega T_3 \end{array} \qquad \begin{array}{lll} T_1 & \omega^2 T_2 & \omega T_3 \\ \omega T_1 & T_2 & \omega^2 T_3 \\ \omega^2 T_1 & \omega T_2 & T_3, \end{array}$$

and they group themselves (by threes) as indicated. Each group does not contain three different types of triangles, but the same triangle affected by rotation through an angle of 120° or 240°. Also the arbitrary turn K causes an arbitrary rotation of the triangle. Therefore, of the nine sets of T's there can be formed only three different kinds of triangles. Hence, if a, b, and c are chosen as real numbers of absolute values less than or equal to 1 to satisfy condition (A), then there are three sets of T's which will give three and only three essentially different metric figures. Any one of these triangles can be obtained from any other by holding one vertex fixed and rotating the second and third vertices around the circumcircle through 120° and 240° respectively.

6. The amount of arbitrariness of a, b, and c will next be considered. Condition (A) represents, in fact, eight conditions, some one of which must be satisfied for a fixed choice of signs. If given $|a| \leqq 1$, $|b| \leqq 1$ (both real) and if the signs of $\pm\sqrt{[1-a^2]}$ and $\pm\sqrt{[1-b^2]}$ are arbitrarily chosen, then (A) may be written

$$(B) \qquad (-a+iA)(-b+iB)(-c\pm i\sqrt{[1-c^2]}) = 1,$$

where, as earlier, A and B each includes the sign of the radical. Regrouping (B), squaring and solving for c, we obtain $c=AB-ab$, which is a real number. Checking this value of c in (B) it is found that the equation is satisfied only if the $+$ sign is chosen in the last parenthesis. Hence, a value of c has been determined which will satisfy some one of the eight conditions in (A). If A and B had signs which are opposite to those which were assigned them, c would still be $AB-ab$, and this value will satisfy one of the eight conditions. Again suppose the square roots are chosen as A and $-B$ (or $-A$ and B); then $c=-AB-ab$. This value will likewise satisfy one of the eight conditions. It can be shown that both of these values of c are of absolute value less than or equal to 1. Hence, given a and b, both real and of absolute value less than or equal to 1, there can be found two values of c, namely $c_1=AB-ab$ and $c_2=-AB-ab$, such that the two sets abc_1 and abc_2 each satisfy two of the eight conditions in (A).

If a and b are not distinct but both equal to u, then $c=-1$, $1-2u^2$. If $a=b=c=u$, then $u=-1$, $1/2$. When $u=-1$, either there is no triangle or the

triangle is equilateral. If $u=1/2$, then condition (A) requires that the signs of the radicals be the same. Whence, from §5 we get the same deductions as when $u=-1$.

Summary. If in equations (4) the three turns T_1, T_2, and T_3 are given, then the three real numbers a, b, and c will be uniquely determined. If, however, three arbitrary real values are given to a, b, and c, it will not in general be possible to find three turns satisfying equations (4). But if the given a, b, and c satisfy the relations

$$|a| \leqq 1, \ |b| \leqq 1, \ |c| \leqq 1;$$

$$(C) \qquad [-a \pm i\sqrt{(1-a^2)}][-b \pm i\sqrt{(1-b^2)}][-c \pm i\sqrt{(1-c^2)}] = 1,$$

then and only then can values of the T's be found, and in this case three homogeneously distinct sets of turn values can be found.

Geometrically, this means if we start with a metric figure, such a metric figure will correspond to a fixed set of T's and for the projective coordinates of the points of that figure, or the points of any figure projective with it, we could use the coordinates of §XVII in which a, b, and c would have their values uniquely determined. Since these values must necessarily satisfy conditions (C), they give rise to three sets of T's, the original set and two others, and the metric figures determined by these three sets are projective with each other. On the other hand, if one gives to a, b, and c arbitrary values, the figure whose points have the coordinates of §XVII will not in general be projective with the metric figure. But, if a, b, and c are given values satisfying conditions (C), then the figure whose points have the coordinates of §XVII will be projective with the metric figure, in fact with three metric figures which are essentially different.

ON MATHEMATICAL LIFE IN HUNGARY[1]

By TIBOR RADÓ, Ohio State University

Hungary has turned out, in the course of the last fifty years, a surprisingly large number of fine mathematicians. The time allotted for this talk would certainly not allow me to give any adequate idea of their contributions to our science; so I have thought I had better not try the impossible. Instead of attempting to sketch what Hungarian mathematicians have done in the way of research, I shall restrict myself to speak of what they are doing in the way of taking care of the next mathematical generation—certainly a business of major importance. In this respect, one thing which is likely to interest you is the *Eötvös-prize.*

This prize was founded in 1894 by the Hungarian Mathematical Society, to commemorate the elevation to the Ministry of Education of the President

[1] A paper read at the Cleveland Meeting of the Mathematical Association of America, on January 1, 1931, by invitation of the program committee.

of the Society, the famous Hungarian physicist, Baron Eötvös. The prize, or more exactly the first and the second prize and one or more honorable mentions, are awarded to the winners of the national mathematical competition for students, held in October every year, and open to the young folks who graduated from the secondary schools in June of the same year. The good mathematicians among these young folks are in October, generally speaking, either at some University, studying mathematics and physics in order to obtain, after four years of study, a secondary teachers diploma, or they are at the Polytechnical School in Budapest. As a matter of fact, several prominent Hungarian mathematicians started studying engineering, and the winner of the Eötvös-prize has many times been a student of the Polytechnical School.

So every October, since 1894, the national competition is entered by ambitious and hopeful baby-mathematicians, of an average age of eighteen years. The competition is sponsored by the Hungarian Mathematical Society. The competitors get three problems and four hours to solve them; they are allowed to use books and everything they want to; the problems are selected, however, in such a way that practically nothing, save one's own brains, can be of any help. As a matter of fact, it does not even help one to have been a real good boy or girl and to have learned all one's formulas and logarithms. The prize is not intended for the good boy; it is intended for the future creative mathematician.

Is it possible to test by problems creative mathematical ability? Certainly it would be possible to prove the impossibility of such an attempt on the basis of psychological theories. First of all, a boy can be really able, and still be unable to solve anything in four hours, in strange surroundings. There is the great danger of discouraging this type. There are many other objections; it might be interesting, but certainly futile, to discuss them, simply because the experience of almost forty years shows conclusively that this mathematical competition *could* be made a success.

Recently, Professor Kürschák, the most enthusiastic promoter of the idea, published in one volume all the problems of the past competitions, with the solutions and a record of the first and second Eötvös prize winners. This book is exceedingly delightful reading; and besides it makes it possible to analyze the working of the Eötvös prize. In order to give you an idea of its working, I pick out the year 1897, and I shall tell you what happened then.

In 1897, one of the three problems was the following.

Prove that the angles α, β, γ of any triangle satisfy the inequality

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma < \tfrac{1}{4}.$$

A first noteworthy feature of the problem is that it certainly can be clearly understood by anybody who has graduated from any secondary school. The notions involved are absolutely familiar. The type of the problem is unusual; indeed, in the secondary schools mathematics consists mostly of identities and computations, whilst here we have got to prove an inequality. Of course, secondary teaching may vary with time and country; I suppose, however, that if

inequalities of the above type had been drilled in Hungarian secondary schools around the year 1897, the above problem certainly would not have been selected for the competition.

According to the book of Professor Kürschák, this problem has been attacked by the first and the second prize winners in the following ways.

Solution by the boy b_1. (I am sorry to say, no girl g ever won the Eötvös-prize.) We have

$$\tfrac{1}{2}\alpha = 90^\circ - \tfrac{1}{2}(\beta + \gamma) < (90^\circ - \tfrac{1}{2}\beta) < 90^\circ.$$

For angles less than 90°, the sine increases with the angle; so it follows that

$$\sin \tfrac{1}{2}\alpha < \sin (90^\circ - \tfrac{1}{2}\beta) = \cos \tfrac{1}{2}\beta$$

and

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta < \cos \tfrac{1}{2}\beta \sin \tfrac{1}{2}\beta = \tfrac{1}{2} \sin \beta \leqq \tfrac{1}{2}. \tag{1}$$

Suppose the notation has been chosen in such a way that γ is the *smallest* angle of the triangle; then $\gamma \leqq 60^\circ$ and

$$\sin \tfrac{1}{2}\gamma \leqq \sin 30^\circ = \tfrac{1}{2}. \tag{2}$$

So it follows from (1) and (2) that

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma < \tfrac{1}{4}.$$

Solution by the boy b_2. This fellow remembered, first of all, quite a lot of his plane geometry. He started with the formula

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma = \rho/4r \tag{3}$$

where ρ is the radius of the inscribed and r the radius of the circumscribed circle. As evidently $\rho < r$, the boy b_2 obtained, without any computations, the desired inequality

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma < \tfrac{1}{4}.$$

The boy b_2 then did a very curious thing. Instead of enjoying the nice solution he had found, he started wondering if the result were as good as possible; and he asked the question if the bound, $\frac{1}{4}$, could not be lowered. And, indeed, he completed his solution by showing that

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \leqq \tfrac{1}{8},$$

and that $\frac{1}{8}$ cannot be replaced by any smaller number. To show this, he refers to the formula of Euler

$$d^2 = r^2 - 2\rho r, \tag{4}$$

where ρ, r have the previous meaning, and d is the distance of the centers of the inscribed and of the circumscribed circles. From (4) it follows that $r^2 - 2\rho r \geqq 0$, and therefore that $\rho/r \leqq \frac{1}{2}$. Substituting into (3), we get

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \leqq \tfrac{1}{8};$$

and, as for the equilateral triangle ($\alpha=\beta=\gamma=60°$) the left-hand side is exactly equal to $\frac{1}{8}$, the constant $\frac{1}{8}$ cannot be improved any more.

You may be impressed also by the fact that the boy b_2 remembered the formula (4) of Euler. I suppose that back in 1897 the geometry of the triangle had been much more attended to in secondary schools than in these days, when the young people get calculus in the upper grades. So it may not be astonishing at all that the boy b_2 remembered that formula which in these days the students possibly never hear of. Really impressive, however, is the fact that the boy b_2 showed what we might call *the pioneer spirit.* He arrived at the proposed goal all right, but he kept on going until he satisfied himself that it was certainly impossible to go any further. This pioneer spirit makes the scientist; it makes, as a matter of fact, the leaders in all fields of human activity.

You will not be very much surprised if I tell you that the boy b_2 has developed into a mathematician of international fame. However, before discussing the 1897 situation any further, I want to give a third solution of the above problem, due to boy B. He is denoted by B, because he was some grown-up mathematician; and indeed, the picture would be very incomplete if I would not tell something concerning the general interest which is given to this competition by the adult Hungarian mathematicians. They try to solve the problems as nicely as possible, and mainly as psychologically as possible. That is to say, they try to solve the problems as a boy who is gifted but who did not waste any too much time on learning things would handle the question. In this respect, the brilliant solution of the boy b_2 is open to the psychological objection that a gifted, but very lazy boy would not remember all the formulas the boy b_2 was using. So some grown-up mathematician proposed the following solution of the problem. We have

$$\cos (\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta) = \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta - \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta,$$

$$\cos (\tfrac{1}{2}\alpha - \tfrac{1}{2}\beta) = \cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta + \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta.$$

Subtraction gives

$$2 \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta = \cos (\tfrac{1}{2}\alpha - \tfrac{1}{2}\beta) - \cos (\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta). \tag{5}$$

Now, on account of the relation $\alpha+\beta+\gamma=\pi$,

$$\cos (\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta) = \sin \tfrac{1}{2}\gamma.$$

Besides,

$$\cos (\tfrac{1}{2}\alpha - \tfrac{1}{2}\beta) \leqq 1. \tag{6}$$

So it follows from (5) that

$$2 \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \leqq 1 - \sin \tfrac{1}{2}\gamma.$$

Multiplying by $\sin \frac{1}{2}\gamma$ and using the inequality between the geometric and the arithmetic means, we obtain

$$2 \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \leqq (1 - \sin \tfrac{1}{2}\gamma) \sin \tfrac{1}{2}\gamma \leqq \left(\frac{(1 - \sin \tfrac{1}{2}\gamma) + \sin \tfrac{1}{2}\gamma}{2}\right)^2 = \tfrac{1}{4},$$

that is to say

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma \leqq \tfrac{1}{8}. \tag{7}$$

The sign of equality can only hold here if it holds in (6), that is to say if $\alpha = \beta$. As the notation was arbitrary, this means that *any* two angles of the triangle must be equal, that is to say $\alpha = \beta = \gamma = 60°$. In this case, we have in (7) obviously the sign of equality.

So far we have described one of the 1897 problems. The point is, of course, did the boys b_1 and b_2 fulfill the expectations which the promoters of the Eötvös-prize hoped the winners ought to fulfill? The year 1897 is in a certain way representative in this respect. I mentioned already that the boy b_2, with the pioneer spirit, developed into one of the prides of Hungarian mathematics. As to the boy b_1, I really do not know what became of him—certainly he did not ever do any creative work in mathematics. Viewed as a whole, this situation is characteristic for the almost forty years of the history of the Eötvös-prize; several of the winners are distinguished mathematicians today, and many of them disappeared from the scene completely.

As a matter of fact, no matter what kind of human ability you try to test, it is certain that a test, that is to say an artificial emergency, never can conclusively show what would happen in a real emergency. Making this necessary allowance, it may be fairly said that the Eötvös-prize is a real success. In the long series of the winners since 1894 we certainly see names which never occured again in connection with mathematical research. On the other hand, many of the winners developed into distinguished mathematicians. The important point, however, is that the Eötvös-prize *is* considered as a success in Hungary. I mean that the bright boys are real anxious to win it, and the winner is then helped in all ways to develop his abilities. To show the present-day attitude, the following little anecdote may be helpful. Every year, when the grown-up mathematicians discuss the problems proposed to and the solutions devised by the young competitors for the Eötvös-prize, one of the prides of Hungarian mathematics tells to anybody who might listen the story of why he did *not* get the prize when it was his year. He points out that in October of that year he was in Zürich, studying engineering at the Polytechnical School there, and he also points out that when his friends wrote him about the problems *he solved them in just one half of an hour.*

The great thing about the Eötvös-prize is exactly that it is considered highly desirable by the young folks leaving the secondary schools, and that the grown-up mathematicians feel that a fair percentage of the winners is made up of potential creative mathematicians, and so the winners are taken care of accordingly. This is certainly a very gratifying situation, and the credit for it

goes to people like Professor Kürschák who helped selecting the problems and who kept on watching and stimulating the winners—the crucial points of the whole scheme.

As a newcomer to this country, I could hardly venture suggestions for something analogous here in America. One thing I feel quite sure about is that the story of the Eötvös-prize, as it is presented in the book of Professor Kürschák, would make a highly interesting reading for those in this country who are anxious to maintain and to increase the enthusiasm for our beautiful science.

SOME MATHEMATICAL ASPECTS OF THE NEW PHYSICS[1]

By J. H. VAN VLECK, University of Wisconsin

The most fundamental and far-reaching formulation of the new quantum mechanics is probably that in terms of matrices, along the lines of the Dirac transformation theory. The term "wave mechanics" is a bit misleading, as the wave idea is only a part of the whole story. Nevertheless, for simplicity, attention is confined in the present paper to phenomena capable of description by means of the Schrödinger wave equation. For a system with one degree of freedom, this equation, which is to be regarded as a fundamental postulate, is

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(W - V(x))\psi = 0. \tag{1}$$

Here h is Planck's constant 6.55×10^{-27} ergs $\times$ sec., m is the mass of the particle, $V(x)$ is its potential energy, a function of the position x, and W is its total (i.e. kinetic+potential) energy, a constant.

One immediately wonders how quantized phenomena can be extracted from (1), as a differential equation suggests things continuous rather than discrete. The answer is that one must confine one's attention to what may be called "civilized solutions" of Eq. (1), and this requirement restricts the constant W to certain particular values. I purposely use the term "civilized" instead of some more exact word such as "analytic" because the conditions which it is necessary to impose on ψ are rather less severe than that ψ be analytic, and furthermore vary somewhat with the physical conditions of the problem. It is usually sufficient that ψ be single-valued, continuous together with its first derivatives, and finite except perhaps at a few points at which ψ becomes infinite in such a way that $\int|\psi|^2dx$ exists. Now although Eq. (1) possesses two arbitrary constants of integration and hence an infinity of solutions, it often turns out that with an arbitrary value of W not one of these solutions is civilized. Instead it often happens that a civilized solution is obtained only if W is given certain particu-

[1] A paper given before the Mathematical Association of America at its meeting at the University of Minnesota, September 8, 1931.

lar, discrete values, usually called the "characteristic values." (Eigenwerte). The corresponding civilized solutions ψ are called the "characteristic functions" (Eigenfunktionen). The characteristic values of W, when substituted in the relation

$$h\nu = W' - W'' \tag{2}$$

enable one to calculate the spectral frequencies ν of an atom or molecule. Eq. (2) is called the Bohr frequency condition, and is merely an expression of the conservation of energy. When an atom passes from a state of energy W' to another W'', it must dispose of an amount of mechanical energy $W' - W''$ which is converted into radiant energy; according to the light-quantum hypothesis the energy of a radiation of frequency ν is $h\nu$, yielding immediately Eq. (2). One can immediately understand from (2) why the spectral lines of an atom have definite frequencies, since by (2) the spectral frequencies have only certain particular values whenever this is true of W. However, usually there are certain regions in which the characteristic values of W form a continuous rather than discrete manifold, and this is nicely reflected physically in the fact that atoms possess continuous as well as discrete spectra.

Great success has attended the calculation of the characteristic colors of the various chemical elements, especially the lighter or simpler ones, by means of (2); but instead of pursuing this further I shall outline briefly some of the different mathematical methods which can be used to find the characteristic values and characteristic functions of Eq. (1). These methods may be roughly divided into four classes:

(a) Infantile methods.
(b) Polynomial methods.
(c) Numerical integration.
(d) Power series in a parameter (perturbation theory).

By (a) I mean cases where the appropriate solutions of the differential equation are obvious by inspection. This situation presents no particular mathematical interest, but in order to have a simple concrete illustration of characteristic

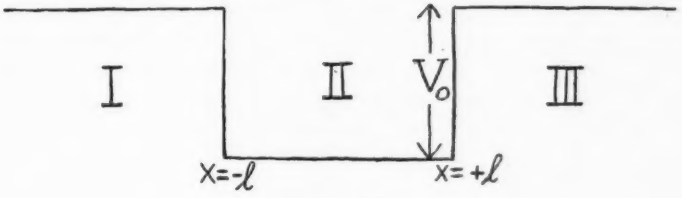


FIG. 1.

functions and values, it is perhaps well to work out an easy problem by this method. Let us therefore consider the problem of a particle in a box extending from $x = -l$ to $x = +l$ (Fig. 1). If we treat the walls as discontinuities in potential energy of amount V_0, then we may take

$$\text{I} \quad V = V_0 \quad (x < -l), \quad \text{II} \quad V = 0 \quad (-l < x < +l), \quad \text{III} \quad V = V_0 \quad (x > l). \quad (3)$$

Here region II represents the interior of the box, and I, III the space outside it. It is convenient to consider separately the differential equation (1) for the regions I, II, III, as there is then great simplification arising from the fact that V has one of the constant values (3) rather than varying with x as in most problems.

It is necessary to consider separately the cases $W < V_0$, and $W > V_0$, which correspond respectively to the particle being unable and able to climb out of the box in classical mechanics. If we introduce (3) and assume $W < V_0$, the solutions of (1) for the regions I–II–III are

$$\text{I} \quad \psi = Ae^{-\alpha x} + Be^{\alpha x}, \qquad \text{II} \quad \psi = C \cos(\beta x - \epsilon), \qquad \text{III} \quad \psi = De^{-\alpha x} + Ee^{\alpha x}$$

where $\alpha^2 = 8\pi^2 m(V_0 - W)/h^2$, $\beta^2 = 8\pi^2 m W/h^2$. To avoid singularities at $x = -\infty$ and $x = +\infty$ we can immediately take $A = E = 0$. In order for ψ to be a civilized solution, it is necessary that both ψ and $d\psi/dx$, and so also $d \log \psi/dx$, be continuous at the boundaries I–II and II–III. From the requirement that this logarithmic derivative be continuous at the boundary I–II one finds

$$\beta \tan(\beta l + \epsilon) = \alpha. \quad (4)$$

The analogous equation for the boundary II–III differs only in that $+\epsilon$ is replaced by $-\epsilon$. This analogous equation is consistent with (4) only if $\epsilon = 0$ or $\epsilon = \frac{1}{2}\pi$, (mod. π). When ϵ is specialized to 0 or $\frac{1}{2}\pi$, and explicit values are substituted for α and β, Eq. (4) becomes a transcendental equation for W. It has a discrete succession of roots W, which are the desired characteristic values. (We have made only $d \log \psi/dx$ continuous, but ψ as well is easily made continuous by proper choice of the amplitudes B, C, D.)

When $W > V_0$, the solutions in regions I and III become trigonometric instead of exponential, and it is no longer necessary to sacrifice some of the arbitrary constants to avoid singularities at $x = \pm\infty$. In consequence one readily finds that ψ, $d\psi/dx$ can be made continuous at the boundaries for all values of W in the interval $W > V_0$. Thus this case furnishes an example of a continuous distribution of characteristic values, to be contrasted with the discrete one in the interval $W < V_0$. If we adopt the physicists' "energy-level diagram," whereby the allowed values of W are represented by the heights of vertical lines, the situation is as represented in (a), Fig. 2, except that in our box problem the valley of potential energy is, of course, rectangular instead of round.

In region II, V equals zero, so that the particle is there entirely free. By comparison of the solution for the region II with the standard expression $\cos(2\pi x/\lambda - \epsilon)$ for a periodic disturbance in terms of wave-length, one sees that a free particle may be regarded as having a wave-length $\lambda = 2\pi/\beta = (h^2/2mW)^{\frac{1}{2}}$. As $W = \frac{1}{2}mv^2$, this may be written $\lambda = h/mv$, which is the well-known formula of the Davisson-Germer effect (see page 95).

(b) As illustrative of the polynomial method we may consider the harmonic

oscillator, i.e. $V=\frac{1}{2}ax^2$. On introducing the following abbreviations and changes of variable

$$A = \frac{4\pi W}{h}\left(\frac{m}{a}\right)^{1/2}, \quad y = \frac{m^{1/4}a^{1/4}(2\pi)^{1/2}}{h^{1/2}}x, \quad \psi = e^{-y^2/2}f,$$

the wave equation (1) takes the form

$$\frac{d^2f}{dy^2} - 2y\frac{df}{dy} + (A-1)f = 0.$$

If we assume a series solution $f=y^k(c_0+c_2y^2+c_4y^4\cdots)$, then from the usual indicial equation it is found that either $k=0$ or $k=1$. The recursion formula for determining the coefficients is

$$\frac{c_{j+2}}{c_j} = \frac{2j+2k+1-A}{(j+k+1)(j+k+2)}.$$

The series will terminate at say $j+k=n$ if $A=2n+1$. In virtue of the definition of A, this demands that

$$W = (n+\tfrac{1}{2})h\nu_0, \quad \text{where} \quad \nu_0 = \frac{1}{2\pi}\left(\frac{a}{m}\right)^{1/2}. \tag{5}$$

Eq. (5) gives the characteristic values for the harmonic oscillator, as the corresponding solutions are highly civilized, viz. an exponential times a polynomial. These are the only characteristic values, for if the series does not terminate it is found that ψ becomes infinite at $x=\pm\infty$.

To use the polynomial method successfully it is necessary that the recursion formula involve only two coefficients. Unfortunately this condition is fulfilled in only a limited number of problems, apparently in every case precisely those problems which are integrable in terms of circular functions in classical mechanics. In most problems it is necessary to resort to an approximate method of solving (1).

(c) Numerical integration is occasionally useful. It sometimes happens that the differential equation possesses two singular points, such that with arbitrary W one can develop an analytic solution about one of these points, and another such solution about the other point. The necessary and sufficient condition that the solution be civilized is that these two developments join on to each other continuously; i.e., represent one and the same solution. It may be possible to meet this condition only if W be given certain particular values, which are the desired characteristic values.

An indirect but powerful method of performing the numerical integration is to minimize the integral whose Euler equation is (1). This method is known as the Ritz method.

(d) Development of the solution as a power series in a parameter is a device

very commonly used. To employ this method it is essential that one term in the potential energy be small, so that the latter can be written:

$$V(x) = V'(x) + \lambda V''(x),$$

where $|V''| \ll |V'|$, and where λ is some constant. It is further necessary that the "unperturbed problem," which is obtained by setting $\lambda = 0$, be capable of exact solution. When these conditions are met, one may develop ψ and W as power series in λ, so that $\psi = \psi_0 + \lambda\psi_1 + \cdots$, $W = W_0 + \lambda W_1 + \cdots$. These developments are substituted in (1) and successive powers of λ equated to zero. For instance, from the coefficients of λ^0 and λ^1 one has respectively

$$\frac{d^2\psi_0}{dx^2} + \frac{8\pi^2 m}{h^2}(W_0 - V'(x))\psi_0 = 0, \quad \frac{d^2\psi_1}{dx^2} + \frac{8\pi^2 m}{h^2}[(W_0 - V')\psi_1 + (W_1 - V'')\psi_0] = 0.$$

The first of these equations may be used to determine ψ_0, W_0, then the second to determine ψ_1, W_1, and so on.

An illustration of perturbation theory is furnished by consideration of the Schrödinger equation for a simple pendulum confined to a plane. Here the moment of inertia I and angle ϕ replace the mass m and linear displacement x in (1), while the potential energy V is $-mgl \cos \phi$. Thus this equation is of the form

$$\frac{d^2\psi}{d\phi^2} + (a + b \cos \phi)\psi = 0. \tag{6}$$

As ψ is a cyclic coordinate, the requirement of single-valuedness demands that $\psi(\phi + 2\pi) = \psi(\phi)$. The determination of the characteristic values and functions of (6) subject to this boundary condition constitutes a well known mathematical problem, the so-called "Mathieu" problem, unfortunately only capable of approximate solution. If $|b| \ll |a|$ one may use a series in $\lambda = b/a$. This amounts to figuring perturbations starting with the free rotator, as there is no potential energy when $b = 0$. If $|b| \gg |a|$, one may expand $\cos \phi$ as a Taylor's series, and if one keeps only the first two terms one has merely the wave equation of the harmonic oscillator, which may now be used as the "unperturbed system" or point of departure for calculating the effect of the higher order terms.

The Mathieu problem described above is perhaps rather hackneyed, but an interesting variant is encountered in connection with the transmission of electrons through a crystal. For our purposes it will suffice to regard the crystal as a fixed lattice of heavy positive ions through which the light, mobile electrons migrate. The potential energy of an electron in the field due to the positive ions is clearly a periodic function of x with period d, where d is the "grating spacing," i.e., the distance between two neighboring positive ions. With V an arbitrary Fourier series in $2\pi x/d$, Eq. (1) is of the form known as Hill's equation, and is too general to be useful. If for simplicity the periodic potential function is taken to be a cosine wave (not a particularly good physical approximation), then $V = A \cos (2\pi x/d)$, and the wave equation (1) now again takes the Mathieu

form (6), if we let ϕ denote the ratio $2\pi x/d$. However, ϕ as thus defined is no longer a cyclic coordinate, since it gives the position along what may be dubbed an "infinite roller-coaster" (cf. (b), Fig. 2) instead of the position in the arc of a pendulum as in the preceding paragraph. Hence one no longer has the boundary condition $\psi(\phi+2\pi)=\psi(\phi)$, and this fact makes the determination of the characteristic values, which has been done by P. Morse and by Brillouin, different than for the ordinary Mathieu problem described agove. The general solution of (6) or of any Hill equation can be shown by group theory or its equivalent to be of the form $e^{a\phi}f(\phi)+e^{-a\phi}f(-\phi)$ where $f(\phi+2\pi)=f(\phi)$; one need no longer demand that $a=2\pi ni$ but instead only that a be a pure imaginary. With this more lenient boundary condition, the characteristic values turn out to fall in

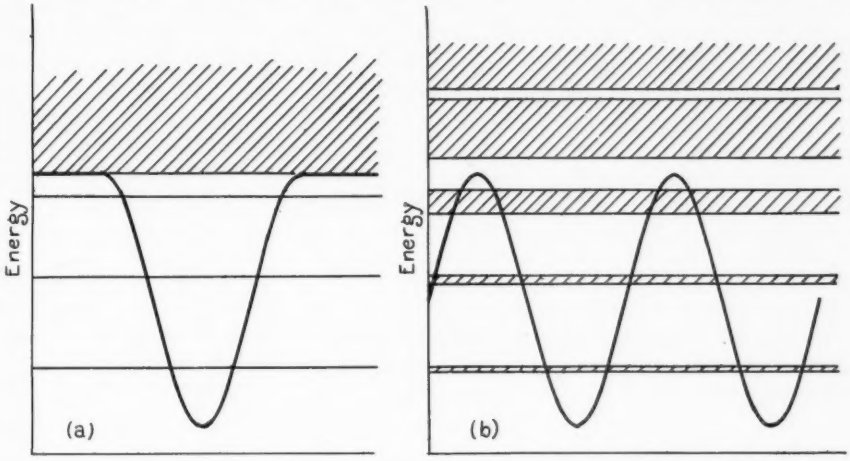


FIG. 2.

continuous manifolds, separated by gaps, as shown in (b), Fig. 2, kindly reproduced from Morse's paper. The situation is to be contrasted with that for a single atom ((a), Fig. 2), where there is only one continuous region (cf. our analysis for the "box" problem) and is confirmed experimentally in that continuous x-ray spectra from crystals are found to contain gaps not observed in similar spectra from monatomic gases.

The results described in the preceding paragraph have an important application to the Davisson-Germer effect, in which crystals are found to reflect electrons of certain particular velocities. Usually this effect is explained by a rather loose optical analogy which appeals to the fact that an electron may be regarded as having a wave length h/mv (cf. p. 92) and that in optics a crystal transmits light of certain wave-lengths and reflects others. Fig. 2, however, shows that a more fundamental explanation can be given in terms of the theory of characteristic values. An electron is capable of transmission through the

crystal ("can ride the roller-coaster") only if its energy falls inside one of the characteristic intervals indicated by dark bands in (b), Fig. 2. If the electron has any other energy or velocity it cannot be transmitted and so has no alternative but to be reflected.

Note that in (b) Fig. 2 some of the bands representing transmission through the crystal fall below the top of the potential wave. This is a manifestation of the fact that in quantum mechanics particles are able to pass through regions of negative kinetic energy and so surmount hills which they could never climb in classical mechanics (a sort of rough analog to the penetration of the geometric shadow in the wave theory of light,—not found in geometrical optics). Conduction in a solid can thus be regarded as electrons playing leap-frog from one valley of potential energy to another. The leaping of classically unsurmountable hills furnishes the long-sought mechanism of radioactive decay. If an alpha particle embedded in a nucleus be assumed to have a crater-shaped potential energy curve, then in quantum mechanics there is always a finite probability that a particle inside the crater may pass over (or perhaps better, through) the rim of the crater and coast out to freedom. In the hands of Gurney, Condon, and Gamow, this model has been extraordinarily fruitful in explaining the empirical relations between the decay periods of the radioactive elements and the velocities of their escaping alpha particles.

References: for an introductory survey of the mathematical problems connected with the Schrödinger wave equation, Condon and Morse's "Quantum Mechanics" (McGraw-Hill) is recommended; for the more advanced matrix transformation theory, Dirac's "Principles of Quantum Mechanics" (Oxford University Press).

ON THE EXPANSION OF AN INTEGRAL OF STIELTJES

By H. S. WALL, Northwestern University

1. In his celebrated memoir on continued fractions,[1] Stieltjes considered the function

$$F(z, \lambda) = \int_0^\infty \frac{(1-\lambda)e^{-zu}du}{e^{(1-\lambda)u} - \lambda},$$

where λ is real and positive, and z is a complex variable. He showed that if $\lambda < 1$:

$$F(z, \lambda) = \sum_1^\infty \frac{(1-\lambda)\lambda^{n-1}}{z + n(1-\lambda)}, \tag{1}$$

[1] Stieltjes: *Recherches sur les fractions continues*, Annales de Toulouse, vol. 8, J, pp. 1–122, and vol. 9, A, pp. 1–47, 1894–95. Also published in vol. 32 of the Mémoires présentés à l'Académie des Sciences de l'Institut National de France, and in Stieltjes's collected works.

and when $\lambda > 1$:

$$F(z, \lambda) = \sum_1^\infty \frac{(\lambda - 1)\lambda^{-n}}{z + (n-1)(\lambda - 1)}, \tag{2}$$

while if $\lambda = 1$:

$$F(z, \lambda) = \int_0^\infty e^{-u} du/(z + u). \tag{3}$$

Hence if

$$\phi(\omega, \mu) = 1 + \frac{\omega}{1+\mu} + \frac{\omega^2}{1+2\mu} + \cdots, \tag{4}$$

we have, in the first case:

$$\phi(\omega, \mu) = 1 + \omega\mu^{-1} F(z, \lambda), \lambda < 1, \omega = \lambda, \mu = (1-\lambda)/z, \tag{5}$$

and in the second case:

$$\phi(\omega, \mu) = \omega^{-1}\mu^{-1} F(z, \lambda), \lambda > 1, \omega = \lambda^{-1}, \mu = (\lambda - 1)/z. \tag{6}$$

The series (4) was used by Poincaré in *Les Méthodes nouvelles de la mécanique céleste*, tome II, page 3.

In the following article we will discuss certain arithmetic properties of the coefficients in the asymptotic expansions, $P(1/z)$, of $F(z, \lambda)$ and other related functions.

2. *The asymptotic series for* $F(z, \lambda)$. If we set

$$X = (1-\lambda)/(e^{(1-\lambda)u} - \lambda),$$

and differentiate X, k times in succession, with respect to u we will obtain:

$$d^k X/du^k = (-1)^k X^{k+1} c_k(\lambda, t), t = e^{(1-\lambda)u}, \tag{7}$$

where

$$c_k(\lambda, t) = a_{k,0} t^k + a_{k,1} t^{k-1}\lambda + \cdots + a_{k,k-1} t\lambda^{k-1}.$$

Hence if

$$c_k \equiv c_k(\lambda) = a_{k,0} + a_{k,1}\lambda + \cdots + a_{k,k-1}\lambda^{k-1},$$

we have:

$$X = c_0 - c_1 u + \frac{1}{2!} c_2 u^2 - \cdots,$$

and

$$F(z, \lambda) \sim \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots.$$

On differentiating both members of (7) with respect to u, equating the value of $d^{k+1}X/du^{k+1}$ so found to that obtained by replacing k by $k+1$ in (7), and then setting $u=0$, one may equate coefficients of like powers of λ and obtain the fundamental recursion formula[1] for the $a_{k,i}$:

$$(8) \qquad a_{k+1,i} = (i+1)a_{k,i} + (k-i+1)a_{k,i-1},\ a_{k,0} = 1,\ a_{i,j} = 0 \quad \text{if } j > i-1 \quad \text{or} \quad \text{if } j < 0.$$

Using (8), the following relation between the polynomials $c_k(\lambda)$ readily results:

$$(9) \qquad c_k = (1-\lambda)\lambda \frac{dc_{k-1}}{d\lambda} + (1+[k-1]\lambda)c_{k-1},\ c_0 = 1.$$

The relation (8) or (9) may be used to compute the c_k. The latter may be used to study the location and character of the roots of these polynomials. Since what follows does not depend on these considerations we will merely state the results. *The roots of c_k are all real, negative, simple, and are separated by those of c_{k-1}.*[2]

Explicit formulas for the $a_{k,i}$ may be obtained as follows. By (1):

$$c_k = \sum_1^\infty i^k(1-\lambda)^{k+1}\lambda^{i-1},\ (\lambda < 1).$$

But

$$(1-\lambda)^{k+1} = 1 - \binom{k+1}{1}\lambda + \binom{k+1}{2}\lambda^2 - \binom{k+1}{3}\lambda^3 + \cdots + \binom{k+1}{k+1}\lambda^{k+1},$$

and consequently:

$$\begin{aligned} c_k &= a_{k,0} + a_{k,1}\lambda + a_{k,2}\lambda^2 + \cdots + a_{k,k-1}\lambda^{k-1}, \\ &= 1^k \\ &\quad + \left[2^k - \binom{k+1}{1}1^k\right]\lambda \\ &\quad + \left[3^k - \binom{k+1}{1}2^k + \binom{k+1}{2}1^k\right]\lambda^2 + \cdots, \end{aligned}$$

and therefore:

$$(10) \quad a_{k,i} = (i+1)^k - \binom{k+1}{1}i^k + \binom{k+1}{2}(i-1)^k - \cdots + (-1)^i\binom{k+1}{i}1^k.$$

[1] These numbers occur elsewhere in the literature. See, for example, Chr. Zeller: Bulletin des Sciences Mathématiques, ser. 2, vol. 5, p. 195; E. Locchi: Monatshefte für Mathematik, vol. 4 (1893), p. 85.

[2] By equation (14) §4, the c_k are reciprocal polynomials.

The integral (3) is that of Laguerre and, as is well known, has the formal power series development

$$\frac{1}{z} - \frac{1!}{z^2} + \frac{2!}{z^3} - \frac{3!}{z^4} + \cdots . \tag{11}$$

From (11) it follows that

$$c_n(1) = a_{n,0} + a_{n,1} + \cdots + a_{n,n-1} = n!. \tag{12}$$

Combining (10) and (12) we obtain the curious formula

$$k! = \sum_{i=0}^{k-1} (-1)^i \binom{k+1}{i} S^k_{k-i}, \qquad \text{where} \qquad S^k_{k-i} = 1^k + 2^k + \cdots + (k-i)^k .$$

3. Since the symmetric group $G_{n!}$ contains $n!$ elements, equation (12) suggests that we try to separate the elements of $G_{n!}$ into n sets, $K_{n,p}$, $p=0, 1, 2, 3, \cdots, n-1$, containing $a_{n,p}$ elements, respectively, and such that $K_{n,p}$ shall have certain distinguishing properties. To do that we write out the possible linear permutations of the natural numbers $1, 2, \cdots, n$ for the cases $n=2, 3, 4$, and note that in these three cases the following theorem is true.

THEOREM 1: *The number of distinct linear permutations*:

$$i_1, i_2, i_3, \cdots, i_n,$$

of the first n natural numbers, having the property that just p inequalities:

$$i_k < i_{k-1} \tag{13}$$

hold, is equal to $a_{n,p}$.

Assuming the theorem true for $n=N$, we will prove its validity for $n=N+1$, and since it is obviously true for $n=2$, its truth for all $n>2$ will follow. When (13) holds we will say that there is an adjacent inversion in the permutation. Then, according to our assumption, the numbers of permutations of $1, 2, 3, \cdots, N$ having just p and $p-1$ adjacent inversions are $a_{N,p}$ and $a_{N,p-1}$, respectively. When the number $N+1$ is adjoined to the set, the total number of permutations is multiplied by $N+1$. We may think of $N+1$ new permutations of $N+1$ numbers being made from any permutation P of N numbers by inserting $N+1$ successively between adjacent numbers of P, and preceeding and following the numbers of P. Let Q denote the number of permutations of $N+1$ numbers having just p adjacent inversions. Then clearly nothing is contributed to the number Q by permutations P having less than $p-1$ or more than p adjacent inversions. If P contains just p adjacent inversions, then on inserting $N+1$ between two successive numbers $r>s$, the number of adjacent inversions remains the same. There are p such insertions possible. One more desired permutation results if $N+1$ follows the numbers of P. This gives, in all, $(p+1)a_{N,p}$ as part of the number Q. Finally if P contains exactly $p-1$ adjacent inversions

then it must contain $N-p$ adjacent permanences. Thus if $N+1$ is inserted between successive numbers $r<s$ an adjacent inversion results. One more is obtained if $N+1$ precedes the numbers of P. In all we then have $(N-p+1)a_{N,p-1}$ contributed in this case. There are no others. Hence

$$Q = (p+1)a_{N,p} + (N-p+1)a_{N,p-1}.$$

Hence by (8):

$$Q = a_{N+1,p}.$$

We now define the sets $K_{n,p}$ of elements of $G_{n!}$.

DEFINITION: *The set $K_{n,p}$ of elements of $G_{n!}$ contains all substitutions*:

$$t = \begin{pmatrix} 1, & 2, & 3, \cdots, & n \\ i_1, & i_2, & i_3, \cdots, & i_n \end{pmatrix},$$

in which the permutation: $i_1, i_2, i_3, \cdots, i_n$ contains just p adjacent inversions.

4. *The sets $K_{n,p}$.* Let

$$T = \begin{pmatrix} 1, & 2, & 3, & \cdots, & n \\ n, & n-1, & n-2, & \cdots, & 1 \end{pmatrix},$$

Then we will prove the following theorem.

THEOREM 2. *The transform of any element in $K_{n,p}$ by T is an element of $K_{n,p}$, that is*:

$$T^{-1}K_{n,p}T = K_{n,p}.$$

Considering $T^{-1}tT$ one observes that k goes into $n-k+1$, then $n-k+1$ into i_{n-k+1}, and then if we suppose i_{n-k+1} replaced by j_{n-k+1}, we have:

$$T^{-1}tT = \begin{pmatrix} 1, & 2, & 3, & \cdots, & n \\ j_n, & j_{n-1}, & j_{n-2}, & \cdots, & j_1 \end{pmatrix}.$$

Now one easily verifies that $i_{n-k} \lesseqgtr i_{n-k+1}$ implies $j_{n-k+1} \lesseqgtr j_{n-k}$, and consequently $j_n, j_{n-1}, j_{n-2}, \cdots, j_1$ contains p adjacent inversions if $i_1, i_2, i_3, \cdots, i_n$ does.

Let

$$T^{-1}t_pT = \bar{t}_p$$

be conjugate to t_p in $K_{n,p}$. Then we separate the elements of $K_{n,p}$ into two classes: $\{e_p\}$, $\{g_p\}$. The elements e_p are such that $\bar{e}_p \neq e_p$, while the $g_p = \bar{g}_p$ are self-conjugate elements with respect to T.

Let

$$t_p = \begin{pmatrix} 1, & 2, & 3, \cdots, & n \\ i_1 & i_2, & i_3, \cdots, & i_n \end{pmatrix},$$

$$s \equiv \begin{pmatrix} i_1, & i_2, & i_3, & \cdots, & i_n \\ i_n, & i_{n-1}, & i_{n-2}, & \cdots, & i_1 \end{pmatrix}.$$

Then clearly $t_{n-p-1}=t_p s$ belongs to $K_{n,n-p-1}$. It follows that

$$a_{n,p} = a_{n,n-p-1}. \tag{14}$$

Let r_p denote the number of elements g_p in $K_{n,p}$. Then we will show that:

$$r_p = r_{n-p-1}. \tag{15}$$

We have:

$$\begin{aligned} s &= (i_1, i_n)(i_2, i_{n-1})(i_3, i_{n-2}) \cdots \\ &= t_p^{-1} T t_p, \end{aligned}$$

since the operation $t_p^{-1}Tt_p$ may be obtained by performing the substitution t_p on the cycles of T. We then have:

$$t_p s = T t_p = t_{n-p-1}, \tag{16}$$

and accordingly:

$$T^{-1}(Tt_p)T = \bar{t}_{n-p-1} = t_p T. \tag{17}$$

Let now $g_p^{(i)}$, $i=1, 2, 3, \cdots, r_p$ be the distinct elements g_p in $K_{n,p}$. Then in $K_{n,n-p-1}$ there will be elements $Tg_p^{(i)}$, $i=1, 2, 3, \cdots, r_p$, and these will be elements g_{n-p-1}, $i=1, 2, 3, \cdots, r_p$, since $Tg_p^{(i)}=g_p^{(i)}T$, $i=1, 2, 3, \cdots, r_p$. These are all distinct, for $Tg_p^{(i)}=Tg_p^{(j)}$, $(i\neq j)$, implies that $g_p^{(i)}=g_p^{(j)}$, contrary to hypothesis. Furthermore, $K_{n,n-p-1}$ cannot contain more elements g than $K_{n,p}$, for on reversing the above argument one would find more than r_p elements g in $K_{n,p}$ Hence we have proved (15).

Obviously the totality of elements g in $G_{n!}$ forms a group, G, a subgroup of $G_{n!}$.

5. *Multiplication relations.* As an immediate consequence of (16), (17) we may state the following theorem.

THEOREM 3. *(i) If g_p and g_q are any elements g in $K_{n,p}$, $K_{n,q}$, respectively, then*

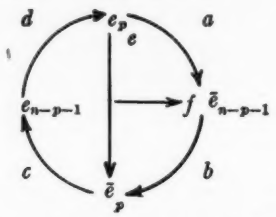
$$g_{n-p-1}g_q = g_p g_{n-q-1}.$$

(ii) Let e_p, $\bar{e}_p$ be any pair of conjugate elements in $K_{n,p}$. Then $e_{n-p-1}=Te_p$, $\bar{e}_{n-p-1}=T\bar{e}_p=e_pT$ is a pair of conjugate elements in $K_{n,n-p-1}$, and the following equations obtain:

(a) $e_p\bar{e}_{n-p-1} = e_p^2 T$,
(b) $\bar{e}_{n-p-1}\bar{e}_p = e_p^2 T$,
(c) $\bar{e}_p e_{n-p-1} = Te_p^2$,
(d) $e_{n-p-1}e_p = Te_p^2$,
(e) $e_p\bar{e}_p = \bar{e}_{n-p-1}^2$,
(f) $e_{n-p-1}\bar{e}_{n-p-1} = \bar{e}_p^2$,

(a_{-1}) $\bar{e}_{n-p-1}e_p = Te_{n-p-1}^2$,
(b_{-1}) $\bar{e}_p\bar{e}_{n-p-1} = e_{n-p-1}^2 T$,
(c_{-1}) $e_{n-p-1}\bar{e}_p = e_{n-p-1}^2 T$,
(d_{-1}) $e_p e_{n-p-1} = Te_{n-p-1}^2$,
(e_{-1}) $\bar{e}_p e_p = e_{n-p-1}^2$,
(f_{-1}) $\bar{e}_{n-p-1}e_{n-p-1} = e_p^2$.

These may be exhibited by the following diagram:



in which the multiplications are carried out in the directions of the arrows in accordance with $a, b, c, \cdots$, *and in the opposite directions in accordance with* $a_{-1}, b_{-1}, c_{-1}, \cdots$.

When $n=4$ we find that the substitutions of G are as follows: $g_0=1$, (in $K_{4,0}$); $g_1=(13)(24)$, $g_1'=(23)$, $g_1''=(1243)$, (in $K_{4,1}$); $g_2=(12)(34)$, $g_2'=(14)$, $g_2''=(1342)$, (in $K_{4,2}$); $T=g_3=(14)(23)$, (in $K_{4,3}$). The elements e are: $e_1=(123)$, $\bar{e}_1=(243)$, $e_2=(142)$, $\bar{e}_2=(134)$; $e_1'=(132)$, $\bar{e}_1'=(234)$, $e_2'=(143)$, $\bar{e}_2'=(124)$; $e_1''=(12)$, $\bar{e}_1''=(34)$, $e_2''=(1423)$, $\bar{e}_2''=(1324)$; $e_1'''=(1432)$, $\bar{e}_1'''=(1234)$, $e_2'''=(13)$, $\bar{e}_2'''=(24)$.

When $n=5$, we find that G is of order 8, just as when $n=4$. The following are the results: $g_0=1$, (in $K_{4,0}$); $g_2=(24)$, $g_2'=(12)(45)$, $g_2''=(1254)$, $g_2'''=(1452)$, $g_2^{(\mathrm{IV})}=(14)(25)$, $g_2^{(\mathrm{V})}=(15)$, (in $K_{5,2}$) ; $T=g_4=(15)(24)$, (in $K_{5,4}$). The sets $K_{5,1}$, $K_{5,3}$ contain no elements of G.

6. *The group G.* We will now prove the following theorem concerning the group G defined in §4.

THEOREM 4. *When* $n=2k$ *or* $2k+1$, G *contains* $2^k k!$ *elements. Denote by* $b_{n,p}$ *the number of elements of* G *contained in* $K_{n,p}$. *Then*

$$
\begin{aligned}
&(a)\quad b_{2k+1,2r+1} = 0,\\
&(b)\quad b_{2k+1,2r} = (2k-2r+1)b_{2k-1,2r-2} + (2r+1)b_{2k-1,2r},\\
&(c)\quad b_{2k,2r} = (2k-2r)b_{2k-2,2r-2} + (2r+1)b_{2k-2,2r} + b_{2k-2,2r-1},\\
&(d)\quad b_{2k+1,2r} = b_{2k,2r-1} + b_{2k,2r},\\
&(e)\quad b_{n,p} = b_{n,n-p-1}.
\end{aligned}
\tag{18}
$$

Note: (a), (b), (c) may be combined to give the single relation

$$(a\,b\,c):\qquad b_{n,p} = (n-p)b_{n-2,p-2} + (p+1)b_{n-2,p} + b_{n-2,p-1},$$

where $p=2r$ if n is odd.

Let $g=(1, i_1)(2, i_2)(3, i_3), \cdots$, be an element of G. Then $T^{-1}gT=g$ implies that $T=(i_1, i_n)(i_2, i_{n-1})(i_3\, i_{n-2}) \cdots$, and hence

$$i_p + i_{n-p+1} = n+1, \qquad (p = 1, 2, 3, \cdots). \tag{19}$$

Now, whether $n=2k$ or $2k+1$, there are just $2^k k!$ permutations $i_1, i_2 \cdots i_n$ satisfying (19). This proves the first part of the theorem.

We now prove the relations (18). Consider a permutation of $2k-1$ numbers:

$$P : i_1, i_2, i_3, \cdots, i_{k-1}, i_k, i_{k+1}, \cdots, i_{2k-1},$$

for which (19) holds. Obviously $i_k = k$. Suppose $i_p \gtrless i_{p+1}$, $(p<k)$. Then by (19), $i_{2k-p-1} \gtrless i_{2k-p}$. It follows that P must contain an even number of adjacent inversions, thus proving (18) (a).

Now suppose the numbers of permutations P having $2r-2$ and $2r$ adjacent inversions are $b_{2k-1,2r-2}$, $b_{2k-1,2r}$, respectively. Then if 0, $2k$ are inserted in P at equal distances from i_k, a permutation of 0, 1, 2, $\cdots$, $2k$ is obtained satisfying (19). If P has $2r-2$ adjacent inversions, it will have $2k-2r$ adjacent permanences. Hence, since if 0 or $2k$ is inserted between $r<s$ an adjacent inversion results, we may obtain $2k-2r$ permutations of $2k+1$ numbers having $2r$ adjacent inversions, from each permutation P. Another such permutation results if $2k$ precedes and 0 follows the numbers of P. Hence there are $(2k-2r+1)b_{2k-1,2r-2}$ obtainable in this manner. If P contains $2r$ adjacent inversions, $(2r+1)b_{2k-1,2r}$ are obtainable. There are no more. This proves (18) (b). The proof of (18) (c) may be carried out in essentially the same way.

To prove (18) (d) we start with the numbers:

$$1, 2, 3, \cdots, k, k+2, \cdots, 2k+1.$$

If we arrange them in some order:

$$i_1, i_2, \cdots, i_k, i_{k+1}, \cdots, i_{2k},$$

satisfying (19), and such that there are just $2r-1$ adjacent inversions, then $i_k > i_{k+1}$, and if $k+1$ is inserted between i_k and i_{k+1}, the resulting permutation must contain $2r$ adjacent inversions. If P contains $2r$ adjacent inversions, then $i_k < i_{k+1}$ and when $k+1$ is inserted between these two numbers, no adjacent inversions are added. Hence we have (18) (d).

In §4 we showed that $K_{n,p}$ and $K_{n,n-p-1}$ contain equal numbers of elements of G. Equation (18) (e) is a restatement of that fact.

7. *The group G and the functions* $F_1(z, \lambda)$, $F_2(z, \lambda)$. If we set:

$$B_{2k}(\lambda) = b_{2k,0} + b_{2k,1}\lambda + \cdots + b_{2k,2k-1}\lambda^{2k-1}, \tag{20}$$

then with the aid of (18) (c) (e), we may verify that:

$$B_{2k} = \lambda(1-\lambda^2)\frac{dB_{2k-2}}{d\lambda} + (1 + \lambda + 2\lambda^2[k-1])B_{2k-2}. \tag{21}$$

Then if:

$$B(x, \lambda) = B_0 + B_2 x + B_4 x^2/2! + \cdots, \quad (B_0 = 1), \tag{22}$$

we will readily see that if $|\lambda|<1$, then $|B_{2k}|<2^k k!$, and consequently the series (22) converges if $|\lambda|<1$, $|x|<\frac{1}{2}$. We may verify by means of (21) that the function $B(x, \lambda)$ satisfies the partial differenial equation:

$$(1 - 2\lambda^2 x)\frac{\partial \Phi}{\partial x} - \lambda(1 - \lambda^2)\frac{\partial \Phi}{\partial \lambda} = (1 + \lambda)\Phi,$$

whence we obtain:

$$\Phi(x, \lambda) = \frac{\lambda - 1}{\lambda}\Psi[x(\lambda^2 - 1) - \log \lambda],$$

where Ψ is an arbitrary function. Let Φ_0 denote that determination of Φ which reduces to 1 when $x = 0$. Then

$$B(x, \lambda) = \Phi_0 = (1 - \lambda)/(e^{(\lambda^2-1)x} - \lambda),$$

and hence if:

$$F_1(z, \lambda) = \int_0^\infty \frac{(1 - \lambda)e^{-zu}du}{e^{(1-\lambda^2)u} - \lambda}, \tag{23}$$

we have the following theorem.

THEOREM 5. *The function $F_1(z, \lambda)$ defined by* (23) *has the asymptotic development:*

$$F_1(z, \lambda) \sim \frac{B_0}{z} - \frac{B_2}{z^2} + \frac{B_4}{z^3} - \cdots,$$

in which the $k+1$th coefficient, B_{2k}, is a polynomial of degree $2k-1$ in λ in which the $p+1$th coefficient, $b_{2k,p}$, is the number of those elements of G, the group of all substitutions on $2k$ numbers which are commutative with $T = (2k, 1)(2k-1, 2)(2k-2, 3)\cdots$, which are contained in the set $K_{2k,p}$.

We may employ Theorem 5 to obtain explicit formulae for the $b_{2k,p}$. In fact if $\lambda > 1$,

$$F_1(z, \lambda) = \sum_{n=1}^{\infty} \frac{(1 - \lambda)\lambda^{n-1}}{z + n(1 - \lambda^2)},$$

and consequently:

$$B_{2k}(\lambda) = (1 - \lambda)\sum_{n=1}^{\infty}\lambda^{n-1}n^k(1 - \lambda^2)^k,$$

$$= 1^k + [2^k - 1^k]\lambda + \left[(3^k - 2^k) - \binom{k}{1}1^k\right]\lambda^2 + \cdots.$$

Hence:

$$b_{2k,p} = \sum_{i=0}^{r}[(p + 1 - 2i)^k - (p - 2i)^k](-1)^i\binom{k}{i}, \text{ if } p = 2r \text{ or } 2r + 1. \tag{24}$$

Now by (18) (d), (24):

$$b_{2k+1,2r} = \sum_{i=0}^{r}(2r+1-2i)^k(-1)^i\binom{k+1}{i}, \tag{25}$$

and if we set:

$$B_{2k+1}(\lambda) = b_{2k+1,0} + b_{2k+1,2}\lambda + \cdots + b_{2k+1,2k}\lambda^k,$$

it may be verified that the following theorem is true:

THEOREM 6. *Set*

$$F_2(z,\lambda) = \int_0^\infty \frac{(1-\lambda)e^{(1-\lambda-z)u}}{e^{2(1-\lambda)u}-\lambda}\,du.$$

Then $F_2(z,\lambda)$ *has the asymptotic development*:

$$F_2(z,\lambda) \sim \frac{B_1}{z} - \frac{B_3}{z^2} + \frac{B_5}{z^3} - \cdots,$$

in which B_{2k+1} *is a polynomial in* λ *of degree* k *in which the* $p+1$*th coefficient is* $b_{2k+1,2p}$, *the number of elements of* G *contained in* $K_{2k+1,2p}$.

In fact,[1] if $\lambda<1$:

$$F_2(z,\lambda) = \sum_{n=1}^{\infty}\frac{(1-\lambda)\lambda^{n-1}}{z+(2n-1)(1-\lambda)},$$

$$B_{2k+1} = \sum_{n=1}^{\infty}(2n-1)^k\lambda^{n-1}(1-\lambda)^{k+1},$$

and we will find that the coefficient of λ^p in this expression is $b_{2k+1,2p}$, given by (25).

8. *Connection with* $\tan x$, $\sec x$. Let E_n, Q_n, denote the number of permutations of $1, 2, 3, \cdots, n$ having an even, and an odd number of adjacent inversions respectively. Then clearly:

$$E_n = [c_n(1) + c_n(-1)]/2, \qquad Q_n = [c_n(1) - c_n(-1)]/2,$$

and,

$$E_n - Q_n = c_n(-1)\begin{cases} = 0 \text{ if } n \text{ is even,} \\ \neq 0 \text{ if } n \text{ is odd.} \end{cases}$$

Now from the equation:

[1] If $\phi_2(\omega, z)$ is the series obtained from (4) by dropping out the 1st, 3rd, 5th, $\cdots$ terms, then ϕ_2 is expressible in terms of $F_2(z,\lambda)$ in a manner analogous to (5), (6).

$$\frac{(1-\lambda)}{e^{(1-\lambda)u}-\lambda} = c_0 - c_1 u + c_2 u^2/2! - \cdots,$$

we will obtain, after certain reductions:

$$\frac{(1-\lambda)\sinh(1-\lambda)u}{1-2\lambda\cosh(1-\lambda)u+\lambda^2} = c_1 u + c_3 u^3/3! + c_5 u^5/5! + \cdots. \tag{26}$$

Setting $\lambda = -1$ in (26), gives:

$$\sinh 2u/(1+\cosh 2u) = c_1(-1)u + c_3(-1)u^3/3! + c_5(-1)u^5/5! + \cdots,$$

from which we find, using the identities $\cosh x = \cos ix$, $\sinh x = -i \sin ix$,

$$\tan x = c_1(-1)x - c_3(-1)x^3/3! + c_5(-1)x^5/5! - \cdots.$$

We then have the following theorem.

THEOREM 7. *Let E_n, Q_n denote the number of permutations of* $1, 2, 3, \cdots, n$ *having an even, respectively, an odd number of adjacent inversions. Then if we set*:

$$A_{2n-1} = (-1)^{n+1}(E_{2n-1} - Q_{2n-1}),$$

we have:

$$\tan x = A_1 x + A_3 x^3/3! + A_5 x^5/5! + \cdots.$$

Let us set

$$\sec x = A_0 + A_2 x^2/2! + A_4 x^4/4! + \cdots.$$

Then it is known that

$$A_{2n} = \sum_{i=0}^{n-1} \binom{2n-1}{2i+1} (-1)^i A_{2n-2i-1},$$

and, consequently, the A_{2n} are expressible in terms of the $E_k - Q_k$.

Let $i_1, i_2, i_3 \cdots i_n$ be a permutation of $1, 2, 3, \cdots, n$ in which the differences in $i_k - i_{k+1}$, $k = 1, 2, 3, \cdots$, are alternately positive and negative. Such a permutation is called an alternating permutation. André[1] has shown that the number $2A_n$ is the number of alternating permutations of $1, 2, 3, \cdots, n$. The connection of these with the Bernoulli numbers is well known.

From Theorem 6 one will readily find that

$$\sec x = B_1(-1) - B_5(-1)x^2/2^2.2! + B_9(-1)x^4/2^4.4! - \cdots,$$

and consequently

$$A_{2n} = (-\tfrac{1}{4})^n B_{4n+1}(-1).$$

We may therefore state the following theorem.

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[1] Desire André: *Sur les permutations alternées*, Journal de Mathématiques (3), 7, 167–184.

THEOREM 8. *Let the numbers of permutations:* $i_1, i_2, i_3, \cdots, i_{4n+1}$, *of* $1, 2, 3, \cdots, 4n+1$, *such that*

$$i_p + i_{4n-p+2} = 4n + 2, \qquad p = 1,2,3, \cdots,$$

(i.e. corresponding to the subgroup G *in* $G_{(4n+1)!}$*), and having a number of adjacent inversions divisible by 4, and not divisible by 4, respectively, be* E'_{4n+1} *and* Q'_{4n+1}. *Set*

$$A_{2n} = (-1/4)^n(E'_{4n+1} - Q'_{4n+1}).$$

Then

$$\sec x = A_0 + A_2x^2/2! + A_4x^4/4! + \cdots, \quad (A_0 = 1).$$

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

NOTE ON SYSTEMS OF ORTHOGONAL CONICS

By B. H. BROWN, Dartmouth College

In a paper entitled "*Sur les courbes orthogonales composées de coniques*," Appell[1] gives the following list of all such systems:

(1) confocal conics
  (a) ellipse-hyperbola
  (b) both families parabolas;
(2) coaxial circles—(4 well-known types, including degenerate cases);
(3) equilateral hyperbolas

$$x^2 - y^2 = \lambda, \quad xy = \mu;$$

(4) ellipses of eccentricity $\frac{1}{2}\sqrt{2}$, and parabolas:

$$2x^2 + y^2 = \lambda, \quad y^2 = \mu x.$$

It is then somewhat disconcerting to find Niewenglowski[2] giving the following orthogonal system of conics

(5) $$(x + 2y)^2 = \lambda(x + y), \quad y = -3x + \mu x^2,$$

a system obviously not included in Appell's list.

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[1] Appell, Grunert, vol. LXIII, 1879, p. 50.

[2] Niewenglowski, *Cours de Géométrie Analytique*, Paris, 1911, vol. 2, p. 318, ex. 5.

The explanation of this apparent discrepancy is simple. Appell assumes "orthogonal" to mean "orthogonal at every finite point of intersection." With this assumption his enumeration is complete. The families given by Niewenglowski have slopes

$$-\frac{x}{3x+2y} \text{ and } \frac{3x+2y}{x},$$

and hence there is orthogonality except when $x=y=0$. In general the parabolas of the first family are tangent to the line $x+y=0$ at the origin, while the parabolas of the second family are tangent to the line $3x+y=0$ at the origin. Thus at the origin, in general, each curve of one family cuts every curve of the other family at $\tan^{-1}\frac{1}{2}$.

The point to be emphasized is that Appell's assumption is a somewhat arbitrary restriction on a problem of some interest, namely: "In a given region, what orthogonal systems of conics exist?" As far as I know this problem is unsolved; it is, apparently, one of considerable difficulty. I have been unable to find any orthogonal systems other than those listed.

The researches of Von der Mühll[1] show that the restriction to *isometric* orthogonal systems is even heavier than that of Appell. His enumeration of all isometric systems of conics does not include either (4) or (5), and consists, with proper choice of parameters, of all the varieties and only those listed under (1), (2), and (3).

A Non-uniformly Convergent Series

By Margaret Gurney, Brown University

It is of some importance, for pedagogical purposes, to have non-artificial examples illustrating the possible difficulties attendant upon a too sanguine application to infinite series of operations which are known to be legitimate for polynomials. Such a one is the following convergent but non-uniformly convergent series

$$\cot z = g'(z) + \frac{1}{z} + \sum_{\nu=1}^{\infty} \frac{2z}{z^2-\nu^2\pi^2} \tag{1}$$

which arises when we try to construct $\sin z$ from the knowledge that its zeros are the points $z=k\pi$ ($k=0, \pm 1, \pm 2, \cdots$). The series (1) is obtained by taking the logarithmic derivative of

$$\sin z = e^{g(z)}\, z \prod_{\nu=1}^{\infty} (1-z^2/(\nu^2\pi^2)), \tag{2}$$

[1] Von der Mühll, *Über die Abbildung von Ebenen auf Ebenen*, Crelle, vol. 69. The actual problem solved by Von der Mühll is somewhat more general: namely, to find all the isometric systems at least one family of which are conics.

in which $g(z)$ is an entire function which is to be determined. It is readily proved that $g'(z)$ is a constant, and the whole question reduces to the determination of this constant.

In (1), if we let z become infinite through pure imaginary values of the variable (i.e. take $z=iy$, where y is real and becomes positively infinite), we have $\cot z=i(e^{-y}+e^{y})/(e^{-y}-e^{y})$, and $\lim_{z\to\infty}\cot z=\lim_{y\to\infty} i(e^{-y}+e^{y})/(e^{-y}-e^{y})=-i$. On the assumption that $\sum_{\nu=1}^{\infty} 2z/(z^2-\nu^2\pi^2)$ is uniformly convergent in the infinite region, we would have

$$\lim_{z\to\infty}\sum_{\nu=1}^{\infty} 2z/(z^2-\nu^2\pi^2)=\sum_{\nu=1}^{\infty}\lim_{z\to\infty} 2z/(z^2-\nu^2\pi^2)=0.$$

This requires that $g'(z)$ be $-i$. But for z real we have $\cot z$ real, and $1/z+\sum_{\nu=1}^{\infty} 2z/(z^2-\nu^2\pi^2)$ real, and we are led to the statement that the difference of two real numbers is an imaginary number. The fallacy comes in the assumption that $\sum_{\nu=1}^{\infty} 2z/(z^2-\nu^2\pi^2)$ is uniformly convergent in the infinite region. That this is not the case can be readily seen by a consideration of the remainder, $R_n=\sum_{\nu=n}^{\infty} 2z/(z^2-\nu^2\pi^2)$. For $z=iy$ we have

$$\frac{i}{2}R_n=\sum_{\nu=n}^{\infty}\frac{y}{y^2+\nu^2\pi^2}\sim\int_n^{\infty}\frac{ydt}{y^2+t^2\pi^2}=\left[\frac{1}{\pi}\tan^{-1}\frac{t\pi}{y}\right]_n^{\infty}$$

$$=\frac{1}{\pi}\left[\frac{\pi}{2}-\tan^{-1}\frac{n\pi}{y}\right].$$

By the proper choice of y (>0), we may make this difference, and therefore $iR_n/2$, anything we like between 0 and $\frac{1}{2}$; for example, for $y=n\pi$, we have $(\pi/2-\tan^{-1}1)/\pi=\frac{1}{4}$. As a result, for a pre-assigned ϵ it is not in general possible to find an M such that, for every $n\geqq M$, we have $|R_n|>\epsilon$, for every z, and the condition for the uniform convergence of $\cot z$ in the infinite region is not fulfilled. The condition is fulfilled, however, for z in the finite region of the plane; and if we allow z to approach 0, we readily obtain $g'(z)=0$, which, in view of (1), leads to the known result $g(z)\equiv 0$.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

College Algebra. By L. J. Rouse. New York, John Wiley and Sons, 1931. x+346 pages. $2.25.

A Treatise on Algebraic Plane Curves. By Julian L. Coolidge. Oxford, The Clarendon Press, 1931. xxiv+514 pages. $10.00.

Philosophy and Modern Science. By Harold T. Davis. Bloomington, Indiana, The Principia Press, 1931. xvi+336 pages. $3.50.

The Constitution of Atomic Nuclei and Radioactivity. By G. Gamow. International Series of Monographs in Physics. Oxford, The Clarendon Press, 1931. 114 pages. $3.50.

Vector and Tensor Analysis. By A. P. Wills. New York, Prentice-Hall, Inc., 1931. xxxii+286 pages. $5.00.

The Foundations of Mathematics. By F. P. Ramsey. New York, Harcourt, Brace, & Co., 1931. xviii+292 pages.

Mathematics. A Textbook for Technical Students. By B. B. Low. New York, Longmans Green, 1931. vi+448 pages.

The Taylor Series. An Introduction to the Theory of Functions of a Complex Variable by P. Dienes. Oxford, The Clarendon Press, 1931. With Bibliography. x+532 pages. $7.50.

Elements of the Theory of Integers. By Joseph Bowden. Garden City, N. Y., Published by the Author, 1931. x+268 pages. $2.50.

The Queen of the Sciences. By E. T. Bell. Century of Progress Series. Baltimore, Williams and Wilkins, 1931. 138 pages. $1.00.

Vectorial Mechanics. By Louis Brand. New York, John Wiley and Sons, 1930. xviii+544 pages. $5.00.

Foundations of the Theory of Algebraic Numbers. By Harris Hancock. Volume I, Introduction to the General Theory. New York, The Macmillan Company, 1931. xx+604 pages. $8.00.

The Theory of Spherical and Ellipsoidal Harmonics. By E. W. Hobson. Cambridge University Press, 1931. xii+500 pages. $11.50.

The Emergence of Life, being a treatise on Mathematical Philosophy and Symbolic Logic by which a new theory of space and time is evolved. By J. B. Burke. Oxford University Press, 1931. 396 pages. $7.50.

Brief College Algebra. By W. L. Hart. New York, D. C. Heath, 1932. viii+334 pages; answers 30 pages. $1.96.

REVIEWS

Introduction to Vector Analysis. By L. R. Shorter. London, Macmillan & Co. 1930. xiv+356 pages.

This is an interesting and useful book and constitutes a unique addition to the literature of this important subject. Its appearance will be welcomed particularly by those who, not having had the advantage of systematic instruction in Vector Analysis, are forced to acquire the desired knowledge through their individual efforts. Of such there are many. Among them will be found those who were in the mind of Professor Kelland when he wrote in his preface to Introduction to Quaternions: "Hamilton and Tait write for mathematicians and they do well, but the time has come when it behooves someone to write for those who desire to become mathematicians."

The plan of Mr. Shorter's book is admirably adapted to the needs of such persons. The proofs of theorems and the solutions of problems are printed in such detail that one reads them as one would a story.

The author has shown even further consideration for the reader, for, whenever reference is made to a formula previously deduced, that formula is repeated in full in the immediate text, thus obviating the necessity of frequent references to preceding pages.

The text includes a large variety of examples of which about a hundred are taken with Dr. Coffin's permission from his Vector Analysis. All of the examples are fully worked out. In this respect we note a departure from the common practice of including a list of unworked problems, but the innovation is well advised in a textbook of this sort.

Sir William Rowan Hamilton at the age of seventeen described his method of studying geometry in these words: "I glance at the title of a proposition and then work it, resolved not to assist myself by text or figure until I conquer the difficulty by my own resources. In general I find this very easy—sometimes not. Still I have observed my rule."

With this process obviously in mind Mr. Shorter has wisely collected the statements of his examples in groups by themselves while their solutions or demonstrations appear in later pages, thus making it easy for a student to try his hand at a solution before consulting that offered by the author. The reader of this book should not miss the opportunity to use the material in this way, for as the author himself suggests, "he will get a more thorough knowledge of the subject and he may also have the satisfaction of hitting on methods which are more simple and concise than those in the text."

The use of scalar and vector products as well as the operator "del" often yields results with such startling ease and swiftness that one may be tempted to question the validity of the process. The author has taken pains to set the mind of the reader at rest on this point by showing that all the important formulae can also be established by the processes of the calculus.

In his development of the subject the author carries the reader along by a succession of easy and natural steps until together they have constructed an instrument of analytical research which is both elegant and powerful.

One may doubt the wisdom of introducing (see page 205) the expression $\nabla \boldsymbol{a}$, the juxtaposition of ∇ and a vector, unless the reader is to become better acquainted with dyads and dyadics later in the book. But this notation is soon discarded, though perhaps not before the reader has experienced a disturbing confusion.

The scope of the work may be judged when one knows that the subjects treated lie chiefly in the field of geometry, kinematics and dynamics, and that the last chapter includes a deduction of Euler's dynamical equations, the divergence theorem and the theorem of Stokes.

As might be expected in the first edition of a book of this character there are typographical errors. The reviewer has preserved a list of those noted in the

hope that it may be useful to the author in the preparation of a second edition a demand for which is predicted.

T. C. Esty

Elementargeometrie der Ebene und des Raumes. (Göschen's Lehrbücherei, Band 16.) By Professor Max Zacharias. Berlin, DeGruyter, 1930. 252 pages. Rm. 13.50.

The author follows the example of Euclid, the earliest writer on elementary geometry, in that he has prepared a book which is not intended as an instruction book on the geometry of the plane and of space for young students but is a scientific development of this theory based on Hilbert's axioms (with occasional slight modifications) and consequently is of interest to the mature student who has some acquaintance with the philosophical notions which have been presented by Hilbert in *Grundlagen der Geometrie*, by Klein in *Elementarmathematik vom höheren Standpunkte aus*, by Enriques in *Questioni riguardanti le matematiche elementari*, and others.

The first section treats of the notions of straight line and of plane, the axioms of order, union, congruence, continuity, and parallelism and their consequences. The second section gives the theory of similarity and its applications which beginning with Euclid's theorems on similar triangles include the circle, the continuous division of a line segment, the theorems of Menelaus, Ceva, Pascal, perspectivities, pole and polar relation, and inversion with respect to a circle. In the third section the topic is plane area, a topic not touched upon by Euclid in his thirteen books. The fourth section, Körperlehre, includes the many topics (exceptions noted below) which in the middle ages and in modern times have been added to the original nucleus, viz., stereometry, Euler's polyhedron theorem, regular polygons, volumes, motion, symmetry, geometry on a sphere, and stereographic projection.

At every stage Prof. Zacharias has given valuable and interesting historical notes, in fact has depicted the historical development of the theory with a completeness which makes the work a contribution of considerable importance.

The modern geometry of the triangle and of the tetrahedron and the theory of conic sections are not discussed because of limitation of space.

M. I. Logsdon

The Elementary Theory of Tensors with Applications to Geometry. By T. Y. Thomas. New York, McGraw-Hill Book Co., 1931. ix+122 pages. $2.50.

This attractive introduction into the theory of tensors contains selections from a course on mechanics given during recent years to an undergraduate class at Princeton University. For this reason the stress is laid not so much on the algebraic and analytical side of the theory of tensors as on the application of the tensor theory to simple problems in kinematics and dynamics. The author shows how vectors and tensors of the second order enter into those questions and how

elegantly they can be solved in tensor notation. The book is therefore particularly adapted to the needs of physics or engineering students who first like to know what you can do with tensors in their field before they start out on a general study.

In many respects the book looks like the common textbook on vectors and dyads translated into the new language of tensor calculus. This has the particular advantage that the student knows from the beginning that vector calculus of his physics and mechanics is only a special case of the tensor calculus of relativity, elasticity, and advanced geometry. It follows the main trend of the times, which seems to be a gradual usurpation of different fields by the symbolism of tensor calculus, as the algebra of invariants (Weitzenböck), the differential geometry of curves and surfaces in ordinary space (Duschek), and elasticity (Nadai).

As particular features of the book we may mention the proof that an arbitrary point transformation which leaves a scalar product of a vector with itself invariant is an orthogonal transformation (p. 29), the subsequent proof that in such case measurement of distance is possible (p. 34), the discussion of simple principles connected with the theory of relativity (p. 42, p. 89), the extensive discussion of the motion of a rigid body, with the screw motion and the theorem of Coriolis (Ch. IV). Chapter V, on Newtonian dynamics, gives Mach's definition of mass, and leads to Lagrange's equations and a number of illustrative examples.

The author has to meet certain difficulties by using tensor notations instead of old vector notations. For the scalar product it is easy enough, it is $v_i w_i$ with the Einstein summation convention. But the vector product offers difficulties. The author meets them by simply writing a new symbol: $C^1 = A^2B^3 - A^3B^2$ (p. 48); this, however, takes away a good deal of the usefulness of the old vector analysis, which is built upon rules connecting the scalar and vector products. Another way is that used by Duschek, who writes $e_{ijk}a_i c_j = c_k$, understanding by e_{ijk} the skew symmetric tensor of value ± 1, 0 (Differentialgeometrie I, p. 31). Thomas introduces this symbol (p. 50), but does scarcely anything with it.

The result is that at the place where the vector product has to be used, namely, in the motion of a rigid body, we get an alternating tensor instead (p. 75); which perhaps is not incorrect, but seems rather inconsistent.

This inconsistency is probably due to the fact that the classification of tensors according to their groups and the relation of vector analysis to tensor analysis is not clearly indicated. On p. 40 the author correctly points out that certain quantities, as the vector product, only have a meaning under the rotational group (Thomas calls it the orthogonal group, but it seems customary to use the word orthogonal for the group of rotations and reflexions; see, e.g., Pascal's Repertorium I, 1, p. 131). But he does not mention clearly that under this group there is no longer any difference between covariant and contravariant quantities. As a result he works with the general affine tensors in the problem of the rotation of a body, which is a problem belonging to the rotational group,

and as a result gets the vector of rotation in the form of a mixed tensor of second order obeying the curious condition (p. 75).

$$w_k^i + w_i^k = 0,$$

which indeed has an invariant meaning under the rotational group, but only here. But there it should better be written with lower indices, while w_{ik} should be from the beginning identified with a vector. This is done only occasionally at the end (p. 76). We therefore think that the relation of ordinary vector analysis to tensor analysis might have been more clearly indicated.

This criticism, however, does not affect the general usefulness of this little book, which is probably the best text we have at present for beginners, especially for students in the applied sciences.

D. J. STRUIK

A Critical Study of the Teaching of Elementary College Mathematics. By Joseph Seidlin. New York, Teachers College Bureau of Publications, 1931. x+108 pages. $1.75.

This book is announced as "a pioneer investigation of actual classroom procedures employed by teachers of college mathematics in twenty Eastern colleges and universities. An attempt is made to summarize, criticize, and evaluate present-day methods of teaching, which, explicitly or by implication, suggest the need for, and possible ways of improving the teaching of elementary college mathematics."

In the opinion of the reviewer, Dr. Seidlin has succeeded admirably in carrying out this professed purpose, and has produced a book which will be valuable to all college teachers.

The first part of the book is a report on the author's visits to 150 college classes, where he was accustomed to record stenographically the actual proceedings. He soon found that all classes could be classified into seven categories: the pure lecture, the lecture with questions and interruptions, the recitation at which time is divided between instructor and students with spontaneous questions by the latter, the combination of blackboard and oral recitation, the question and answer development, the ground-covering, textbook-repeating mechanical recitation, and the simple blackboard recitation. He reports in extenso one or more actual classes of each type, with pertinent criticisms. It comes as a shock to one who has been accustomed to think of the two last-named forms of recitation as rather low and comparatively useless types of activity, to learn that of the 150 classes visited, 106, or more than 70%, were of these types. Apparently the greater part of our college teaching, in the more elementary courses, is carried on in the recitation style which one usually associates with the high school. Dr. Seidlin later in the book evaluates the different types of recitation, and arrives at the conclusion that the two methods in question are the least valuable of the seven. He also shows an interesting relation between the

type of recitation which predominates at each institution, and the percentage of students who take major work in mathematics. The latter figure shows a clear inverse relationship to the prevalence of the purely mechanical recitation forms.

The second chapter is devoted to "Violations of Some Principles of the Learning Process," with full quotations from the author's record of classes he has attended. Every teacher of mathematics will read this chapter with much searching of the heart. With this chapter may be associated a later one, in which are quoted several instances of serious errors of instructors, due either to ignorance or to inexcusable carelessness.

There is a chapter based largely on interviews with heads of departments, dealing with the essential qualities of good teaching and the factors on which it depends. Another chapter shows by numerous quotations how frequently we instructors require certain things because either "It is in the textbook; the author says so;" etc., or because "it will be included in the examination."

There is an evaluation of attempts to improve teaching, and a final chapter of Summary and Conclusions. It would be well if these could be quoted verbatim, and it is hoped that many teachers will be able to read the book. It is not written in the technical jargon of pedagogy, but is a serious, well-considered study of many questions that are of great importance in the minds of all good college teachers.

ROGER A. JOHNSON

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3528. *Proposed by Albert A. Bennett, Brown University.*

If a, b, c, be complex numbers such that

$$|a| = |b| = |c| = r \neq 0,$$

then $|(ab+bc+ca)/(a+b+c)| = r$. Generalize.

3529. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

A ship is sailing with a speed and direction, v_1; the wind blows apparently (judging by the vane on the mast) in the direction of a vector, a; on changing the

direction and the speed of the ship from v_1 to v_2, the apparent wind is in the direction of a vector, b.

Find the vector velocity of the wind.

3530. *Proposed by Allen A. Shaw, University of Arizona.*

Prove that the product of any three consecutive integers is a multiple of 504, if the middle integer is a perfect cube.

3531. *Proposed by Eugene M. Berry, Lynchburg College.*

We have given a horizontal bar of length l, with a (vertical) leg at each end of the bar. The legs are short and of equal length. The front leg comes to a point at the bottom while the foot of the rear leg is a knife edge which is parallel to the bar. If this is placed with the legs on paper find the curve traced by the knife edge when the front leg traces a given circle.

3532. *Proposed by R. E. Gaines, University of Richmond.*

If the triangle PQR is inscribed in an ellipse so as to cut off three segments of equal area, and if tangents are drawn at P, Q, and R, the triangle thus formed will be inscribed in a second ellipse which will be divided in the same way.

3533. *Proposed by R. E. Gaines, University of Richmond.*

The generating circle of a cycloid in one position is tangent to one arch of the cycloid and intersects the next following arch in P_1 and P_2; another such circle intersects the same arch in P_2 and P_3; another, in P_3 and P_4; and so on indefinitely. If $\alpha_1, \alpha_2, \alpha_3, \cdots$ are the angles which the chords P_1P_2, P_2P_3, $P_3P_4, \cdots$ make with the base of the cycloid, then

$$6\sum_{n=1}^{\infty}\alpha_n = 4\pi - 3^{3/2}.$$

3534. *Proposed by J. V. Uspensky, Stanford University.*

Show that

$$\left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t-1}{t^r-1}\right)\right]_{t=1} = [1-r][1-2r]\cdots[1-(n-1)r]r^{-n}.$$

SOLUTIONS

275. [1917, 467; 1931, 342]. *Proposed by V. M. Spunar, Chicago, Ill.*

A square, side $2a$, is represented by the equation $x^n+y^n=a^n$ $(n=\infty)$. Find a like formula for an equilateral triangle.

Solution by Ralph P. Agnew, Princeton University

This problem may be regarded from two quite different points of view which depend on different interpretations of the word "represents."

From one point of view, we consider the curves C_n having the equations

$x^n+y^n=a^n$, $n=1, 2, 3, \cdots, a>0$. When n is even, it is easy to determine the intersection of C_n and an arbitrary half-ray emanating from the origin, and to show that as n increases through even integral values this intersection approaches the point where the half-ray meets the square of side $2a$. Hence if we let C_∞ represent the square of side $2a$, we may write $\lim_{n\to\infty} C_{2n}=C_\infty$ and use this fact as a basis for the assertion that $x^{2n}+y^{2n}=a^{2n}$, $n=\infty$, "represents" C_∞. When n increases through odd values, C_n does not approach C_∞, but only that part of C_∞ which lies in the half-plane $x+y\geqq 0$. With this interpretation of the term "represents," it is easy to solve the problem at hand. The transformation

$$\begin{cases} x' = x \\ y' = (x+a)y/(3^{1/2}a), \end{cases}$$

which carries the line $y=a$ into the line $y'=(x'+a)/3^{\frac{1}{2}}$, carries the square C_∞ into the equilateral triangle C_∞' whose vertices are $(-a, 0)$, $(a, 2a/3^{\frac{1}{2}})$, and $(a, -2a/3^{\frac{1}{2}})$. This transformation carries the curves $C_n: x^n+y^n=a^n$ into the curves

$$C_n': x'^n + [3^{1/2}ay'/(x'+a)]^n = a^n$$

and it is easily shown that $\lim_{n\to\infty} C_{2n}'=C_\infty'$; hence the equation

$$x^{2n} + [3^{1/2}ay/(x+a)]^{2n} = a^{2n},\ n=\infty,$$

may be said to represent an equilateral triangle.

From another point of view, we may regard ∞ as a member of the real number system which satisfies the conditions $z^\infty=0$ when $-1<z<1$ and $(z^2)^\infty=1$ when $z=\pm 1$. Then the equation $x^{2n}+y^{2n}=a^{2n}$, $n=\infty$, when written in the form

$$(x^2/a^2)^\infty + (y^2/a^2)^\infty = 1,$$

represents the square C_∞ with its vertices omitted; for if $|x|>a$, y is undefined; if $|x|<a$, $y=\pm a$; and if $x=\pm a$, then y may be any number satisfying the condition $-a<y<a$. The assertion that $x^n+y^n=a^n$, $n=\infty$ represents C_∞, as well as the assertion that $(x/a)^n+(y/a)^n=1$, $n=\infty$ represents C_∞, can be criticized; for when $z=-a$, $(z/a)^n$, $n=\infty$, becomes meaningless and the left and lower sides of the square do not appear in the graph. A similar analysis shows that the equation

$$[x^2/a^2]^\infty + [3y^2/(x+a)^2]^\infty = 1$$

represents an equilateral triangle with its vertices omitted.

The first point of view furnishes a general method for solving problems of this character, but scarcely justifies use of the term "represents." The second view does justify use of the term, but fails to give a representation of all the points of the square and triangle.

3478. [1931, 112]. *Proposed by Vladimir F. Ivanoff, San Francisco. California.*

A segment M_0M_n of unit length is divided into equal parts by points M_1, $M_2, \cdots, M_{n-1}$. Another segment M_0C of length $\sqrt{(p/n)}$ (p being a given

constant) is drawn perpendicular to M_0M_n, and C is joined with the points of division. M_1M_2' is drawn parallel and equal to CM_2, $M_2'M_3'$ parallel and equal to CM_3, and finally $M_{n-1}'M_n'$ parallel and equal to CM_n. Find the limit of the locus of the vertices of this broken line as $n\to\infty$, and consequently $M_0C\to 0$.

Solution by Ralph P. Agnew, Princeton University

Further interest is added to this interesting problem, and a more complete notation is obtained, by formulating it as follows: Corresponding to each positive integer n, let $D_1^{(n)}, D_2^{(n)}, D_3^{(n)}, \cdots$ represent the points $(1/n, 0)$, $(2/n, 0)$, $(3/n, 0)$, $\cdots$ of a plane rectangular coordinate system, and let $C\equiv C^{(n)}$ be the point $[0, -(p/n)^{\frac{1}{2}}]$. Let $f_n(x)$ be the function whose graph is obtained by drawing $C^{(n)}D_1^{(n)}$ and then by drawing successively $D_1^{(n)}P_2^{(n)}$ parallel and equal to $C^{(n)}D_2^{(n)}$, $P_2^{(n)}P_3^{(n)}$ parallel and equal to $C^{(n)}D_3^{(n)}$, $P_3^{(n)}P_4^{(n)}$ parallel and equal to $C^{(n)}D_4^{(n)}$, etc. Thus, after we define $P_0^{(n)}\equiv C^{(n)}$ and $P_1^{(n)}\equiv D_1^{(n)}$, the graph of $f_n(x)$ is a broken line which extends indefinitely upward and to the right, its vertices being $P_0^{(n)}, P_1^{(n)}, P_2^{(n)}, \cdots$. We are required to determine the behavior of $f_n(x)$ and its vertices as $n\to\infty$.

On comparing the above formulation with that of the proposer, we see that the difference lies in the fact that we define $P_r^{(n)}$ for $r=0, 1, 2, \cdots$ while the proposer defined $P_r^{(n)}$ only for $r=1, 2, \cdots, n$. Let $f_n^*(x)$ represent the function, whose graph is the broken line $P_1^{(n)}P_2^{(n)}\cdots P_n^{(n)}$, which the proposer offered for consideration. Then $f_n^*(x)=f_n(x)$ for all x for which $f_n^*(x)$ is defined.

For a fixed n, the coordinates of $P_0^{(n)}, P_1^{(n)}, P_2^{(n)}, \cdots$ are easily evaluated; the abscissa and ordinate of $P_r^{(n)}$ are respectively $r(r+1)/(2n)$ and $(r-1)[p/n]^{\frac{1}{2}}$.

Hence

$$f_n\left(\frac{r(r+1)}{2n}\right)=(r-1)\left[\frac{p}{n}\right]^{1/2}.$$

If the integers r and n are increased in such a way that the ratio $r(r+1)/(2n)$ maintains a constant rational value, say ξ, then the expression $(r-1)[p/n]^{\frac{1}{2}}$ tends to the limit $(2p\xi)^{\frac{1}{2}}$; hence we are led to suspect that $f_n(x)\to(2px)^{\frac{1}{2}}$ as $n\to\infty$.

We will now show that $f_n(x)$ converges uniformly over the entire range $x\geqq 0$ to $(2px)^{\frac{1}{2}}$ as $n\to\infty$. Let ξ be any number $\geqq 0$; then, corresponding to each fixed positive integer n, there is an integer $r=r_n(\xi)\geqq 0$ such that

$$\frac{r(r+1)}{2n}\leqq\xi<\frac{(r+1)(r+2)}{2n}.$$

Since $(2px)^{\frac{1}{2}}$ and $f_n(x)$ are both increasing functions of x, we have

$$\left[2p\,\frac{r(r+1)}{2n}\right]^{1/2}\leqq(2p\xi)^{1/2}<\left[2p\,\frac{(r+1)(r+2)}{2n}\right]^{1/2}$$

and

$$f_n\left(\frac{(r+1)(r+2)}{2n}\right) > f_n(\xi) \geqq f_n\left(\frac{r(r+1)}{2n}\right).$$

After subtracting and simplifying, we obtain

$$(p/n)^{1/2}[(r^2+r)^{1/2}-r] < (2p\xi)^{1/2} - f_n(\xi) < (p/n)^{1/2}[(r^2+3r+2)^{1/2}-(r-1)].$$

The function $[(r^2+r)^{\frac{1}{2}}-r]$ is $\geqq 0$ when $r \geqq 0$; and the function $[(r^2+3r+2)^{\frac{1}{2}} - (r-1)]$ is, for $r \geqq 0$, an increasing function of r which $\to 5/2$ as $r \to \infty$. Hence

$$0 < (2p\xi)^{1/2} - f_n(\xi) < (5/2)(p/n)^{1/2}.$$

Since $(p/n)^{\frac{1}{2}} \to 0$ as $n \to \infty$, and since this approach is independent of the choice of ξ the uniform convergence of $f_n(x)$ to $(2px)^{\frac{1}{2}}$ over the interval $x \geqq 0$ is established.

Since the function $f_n(x)$ converges uniformly to $(2px)^{\frac{1}{2}}$, all of the limit points of vertices of the functions $f_n(x)$ must lie on the semi-parabola S whose equation is $y=(2px)^{\frac{1}{2}}$. In conclusion, we will show that each point of S is a limit point of vertices of functions $f_n^*(x)$ and a *fortiori* a limit point of vertices of the functions $f_n(x)$. It is sufficient to show that each point of S with a rational abscissa is such a limit point. Let an arbitrarily chosen positive rational number be written in the form a/b where a and b are positive integers. Then $P_{r(k)}{}^{(n(k))}$, where $r(k)=2ka$ and $n(k)=kb(2ka+1)$, is, for each positive integer k, a vertex of $f_{n(k)}^*(x)$ whose abscissa is a/b; hence as $k \to \infty$ and $n(k) \to \infty$, the vertex $P_{r(k)}{}^{(n(k))}$ approaches the point on S whose abscissa is a/b and the result is established.

A Note by Otto Dunkel. In the first part of the above proof an additional inequality may be written. The segment $P_r{}^{(n)}P_{r+1}{}^{(n)}$, which contains the point $[\xi, f_n(\xi)]$, lies within a vertical strip with the base ends $x_1=\xi-\epsilon$, $x_2=\xi+\epsilon$, where $\epsilon=[(2\xi)^{\frac{1}{2}}+1]n^{-\frac{1}{2}}$. This, with the above inequality, shows that, as $n \to \infty$, $r=r_n(\xi)$, each point of $P_r{}^{(n)}P_{r+1}{}^{(n)}$ approaches ξ as a limit. This provides a second proof of the last part of the solution.

Also solved by A. Pelletier, R. Rosenbaum, F. Underwood, F. L. Wilmer, and the Proposer.

3479. [1931, 170]. *Proposed by C. A. Rasmussen, University of Alabama.*

If B and C be two fixed points and A a variable point at which BC subtends a given angle and if E and F be the feet of the perpendiculars from B and C, respectively, to the opposite sides of the triangle ABC, determine the path of a point P, the intersection of FP with EP, if $\angle BEP$ and $\angle PFC$, respectively, remain equal to $\angle ECB$ and $\angle FBC$.

Solution by Andrew G. Clark, Colorado Agricultural College.

Let θ be the constant angle at A, and $2a$ and O be the length and midpoint, respectively, of the line BC.

From elementary geometry it is known that $EO=FO=a$. Now, a consideration of the angles of the configuration shows that angles PEO and OFP are right angles, that $\angle EOF$ and $\angle EPF$ are supplementary, and that the latter equals 2θ. It follows that angles PEF and EFP are each equal to $90°-\theta$, and hence $EP=FP$.

It now follows at once that EPO and FPO are congruent right triangles, invariant to the variation of the point A except as to position.

Therefore, the point P must describe a circle about O as a center, and with radius equal to $a\csc\theta$.

Editorial Note. In order to remove the uncertainty of the problem as to the construction of P, we may say that P is determined by drawing the angles CFC' and BEB' so that these angles are equal in sense and magnitude to the angles CBA and BCA, respectively. Then P is the intersection of the sides of the angles FC' and EB'. In this way P is defined for the two cases, $\angle BAC=\theta\leqq 90°$ and $\angle BAC=180°-\theta$, where in the second case P is the intersection of $B'E$ and $C'F$ produced. With this definition P traces the entire circumference of the circle found above.

Also solved by L. Green, O. J. Ramler, and F. Underwood.

3483. [1931, 170]. *Proposed by Mabel M. Young, Wellesley College.*

A is a fixed point, B a variable point on a circle with center at O. Show that the locus of the orthocenter of triangle AOB is a strophoid.

Solution by Byron D. Roberts, Parsons College

Choose coordinate axes with origin at O. The equation of the circle is then $x^2+y^2=r^2$. Let the fixed point A be the point $(r, 0)$, and the variable point B have coordinates (x, y). Let (α, β) be the orthocenter.

The obvious relations are then

$$\alpha = x \text{ and } \frac{\beta}{\alpha} = \frac{r-x}{y} = \cot \angle OAB.$$

Eliminate x and y from these two equations and the equation of the circle and get

$$\beta^2 = \alpha^2 \frac{r-\alpha}{r+\alpha}.$$

the equation of the strophoid.

A Note by the Editors. The strophoid is usually defined by means of two straight lines perpendicular at O. From a fixed point A on one a straight line is drawn cutting the other in K, and the point P is taken in two ways on AK so that $KP=KO$. The locus of the points P is the strophoid.

If from A and P perpendiculars are drawn to PO and AO, respectively, they meet in the orthocenter B of AOP. It is easily seen from a figure that PO bisects the angle APB, and hence APB and AOB are isosceles triangles. Thus B lies on the circle with center O and radius OA, and P is the orthocenter of AOB. The other point on AP is given in the same way by the other extremity B' of the diameter BOB'.

Also solved by Leo Aroian, A. G. Clark, Ralph Deutsch, A. E. Gault, R. A. Johnson, L. S. Johnston, Theodore Lindquist, A. Pelletier, R. C. Staley, Hazel Schoonmaker, F. Underwood, Kamcheung Woo, Roscoe Woods, one unsigned solution, and the Proposer.

The following solutions were received after selections were made in September: W. B. Campbell, 3471; Dorothy M. Wilson, 3470; G. A. Williams, 238; D. C. Duncan, 275, and 432; R. P. Agnew, 434.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The American Institute of Physics, an agency of cooperation in the interest of physics, has recently been established by the American Physical Society, the Optical Society of America, the Acoustical Society of America, and the Society of Rheology. This Institute will perfect details of organization and arrange for cooperative publication of physics journals, and assist the newspapers to disseminate accurate news of important developments and applications of physics. Henry Barton has been made the director, and John T. Tate the advisor on publications.

The Henry Burchard Fine Hall of Mathematics of Princeton University was dedicated with appropriate ceremonies on October 22, 1931. On this occasion papers were read by Professors G. D. Birkhoff and G. A. Bliss, both formerly members of the department of mathematics of Princeton.

The Frederick Ives medal of the Optical Society of America has been awarded to Dr. Theodore Lyman, emeritus professor of mathematics and natural philosophy at Harvard.

The John Fritz gold medal, conferred by the four American societies of civil, mining and metallurgical, mechanical, and electrical engineers, has been awarded for 1932 to Professor M. I. Pupin, of Columbia University, for his achievements as "scientist, engineer, author, inventor of the tuning of oscillating circuits and the loading of telephone circuits by inductance coils."

Professor Alonzo Church, of Princeton University, lectured before the Mathematical Clubs of Rutgers University and the New Jersey College for Women, December 4, 1931 on the subject "A Finite Geometry."

Colonel C. P. Echols, professor of mathematics at the United States Military Academy at West Point, has retired from the Army.

Dr. Irwin Roman, mathematical physicist of the Geophysical Research Corporation, has been appointed assistant professor of mathematics at the Michigan College of Mining and Technology.

Professor Elijah Swift, head of the department of mathematics in the College of Arts and Sciences of the University of Vermont, has been made dean of the College.

Professor Charlotte A. Scott, formerly professor of mathematics at Bryn Mawr College, died November 8, 1931, at the age of seventy-three.

Professor G. O. James, of Washington University, St. Louis, died November 24, 1931.

Mr. E. A. Kholodovsky, of the department of mathematics of Columbia University, died in October, 1931.

Professor G. W. Myers, of the University of Chicago, died November 23, 1931.

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period 1929-1931.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of.....Dollars, to be known as the.....Fund, and to be used for[2] { Endowment—the income only of which may be expended. / Special Projects—for which both principal and income may be expended. }

Witness:..... Signature.....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the Secretary.

ILLINOIS, Urbana.
INDIANA.
IOWA.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI, March 11-12.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May.
MICHIGAN, Ann Arbor, March 19.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo.
SOUTHEASTERN, Gainesville, Fla., Mar. 18-19.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE SIXTEENTH ANNUAL MEETING OF THE ASSOCIATION

The sixteenth annual meeting of the Mathematical Association of America was held at New Orleans, Louisiana, on Wednesday and Thursday, December 30 and 31, 1931, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. Two hundred twenty-five were in attendance at the meetings, including the following one hundred thirty members of the Association:

V. W. Adkisson, University of Arkansas
Nola L. Anderson, Sophie Newcomb College
Beulah M. Armstrong, University of Illinois
C. S. Atchison, Washington and Jefferson College

R. W. Babcock, Kansas State Agricultural College
E. A. Bailey, LaGrange College
F. R. Bamforth, Ohio State University
E. T. Bell, California Institute of Technology
B. A. Bernstein, University of California
T. A. Bickerstaff, University of Mississippi
G. D. Birkhoff, Harvard University
Leora Blair, Louisiana State Normal College
G. A. Bliss, University of Chicago
L. M. Blumenthal, Rice Institute
H. E. Bray, Rice Institute
Myrtle C. Brown, State Teachers College, Denton, Texas
H. E. Buchanan, Tulane University

W. D. Cairns, Oberlin College
R. D. Carmichael, University of Illinois
J. W. Cell, Southern Methodist University
L. W. Cohen, University of Kentucky
J. B. Coleman, University of South Carolina
Lennie P. Copeland, Wellesley College
H. M. Cox, Georgia School of Technology
A. T. Craig, University of Iowa
D. R. Curtiss, Northwestern University

R. D. Daugherty, Kansas State Agricultural College
H. T Davis, Indiana University
B. F. Dostal, University of Florida
Otto Dunkel, Washington University
W. L. Duren, Jr., Tulane University
E. P. R. Duval, University of Oklahoma
Nat Edmonson, Jr., Texas Technological College

L. P. Eisenhart, Princeton University
G. C. Evans, Rice Institute
G. W. Evans, Lynn, Mass.

B. F. Finkel, Drury College
C. H. Fischer, University of Iowa
T. M. Focke, Case School of Applied Science
L. R. Ford, Rice Institute
Tomlinson Fort, Lehigh University
Bertha E. Frankenbush, Miller's High School, New Orleans, La.

F. C. Gentry, Tulane University
F. J. Gerst, Loyola University, Chicago, Ill.
J. S. Gold, Bucknell University
Michael Goldberg, Bureau of Ordnance, Navy Dept., Washington, D. C.

W. L. Hart, University of Minnesota
J. O. Hassler, University of Oklahoma
Sarah E. Haughton, University of Albama
E. R. Hedrick, University of California at Los Angeles
Deborah M. Hickey, Delta State Teachers College
Einar Hille, Princeton University
H. M. Hosford, University of Arkansas
Jewell C. Hughes, Hunter College
C. A. Hutchinson, University of Colorado

Louis Ingold, University of Missouri
M. H. Ingraham, University of Wisconsin

Dunham Jackson, University of Minnesota
Rosa L. Jackson, Alabama College
B. W. Jones, Cornell University

F. W. Kokomoor, University of Florida
H. W. Kuhn, Ohio State University

C. G. Latimer, University of Kentucky
F. A. Lewis, University of Alabama
R. G. Lubben, University of Texas